NOTES ON HARMONIC ANALYSIS

XIAOTONG (DAWSON) YANG

Contents

Notation and conventions 2
Introduction 3
1. Fourier Series and Integrals 3
1.1. Introduction to Fourier Series and Differentiation 3
1.2. Convergence of Fourier Series 8
1.3. Summability methods 19
1.4. L^p convergence of Fourier Series 24
1.5. Fourier Transform 25
1.6. Fourier Transform on L^2 29
1.7. Fourier Transform on L^p, $1 < p \leq 2$ 30
1.8. Schwartz Space 33
2. The Hilbert Transform 36
2.1. L^p Convergence of Fourier Transform and the Hilbert Transform 36
2.2. Hilbert Transform Simplification for Fourier Series 40
2.3. Hilbert Transform Simplification for the Fourier Transform 42
2.4. Marcinkiewicz Interpolation Theorem 44
3. Dyadic Intervals 47
3.1. What are Dyadic Intervals? 47
3.2. The Hardy-Littlewood Maximal Function 48
3.3. Calderón-Zygmund decomposition 50
3.4. Weak type estimates for the Hilbert transform 53
3.5. L^p Boundedness of the Hilbert Transform 55
Appendix A. Riesz-Thorin Interpolation 59
References 62

Date: Started June 12, 2025. Last revised July 29, 2025.

Notation and conventions

- Let $x = (x_1, \dots, x_n) \in \Omega \subset \mathbb{R}^n$, and
$$\partial_{x_j} u = \frac{\partial u}{\partial x_j}, \quad j = 1, \dots, n.$$
The gradient of u is
$$\nabla u = (\partial_{x_1} u, \dots, \partial_{x_n} u).$$
- The Laplace operator on $\mathbb{R}^n$ is
$$\Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}.$$
- For any set S, contained in some ambient space X, we denote by χ_S the *indicator function* of S
$$\chi_S : X \to \{0,1\}, \ \ \chi_S(x) = \begin{cases} 1, & x \in S, \\ 0, & x \notin S. \end{cases}$$
- A vector $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ is called a multi-index of order $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We denote
$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}},$$
and similarly for $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$.
- $\overline{\Omega}$ is the closure of Ω, and $\partial\Omega = \overline{\Omega} \setminus \Omega$ is the boundary.
- For Ω bounded, we equip this set with the norm:
$$\|u\|_{C^m(\Omega)} = \sum_{|\alpha| \le m} \sup_{\overline{\Omega}} \|\partial^\alpha u\|_{L^\infty}.$$
This yields a Banach space.
- $C^m(\Omega)$ is the space of m-times differentiable functions, which are continuous up to $\partial\Omega$.
- $C^\infty(\Omega) = \bigcap_{m \in \mathbb{N}} C^m(\Omega)$ is the space of smooth functions.
- $F = \{f : f \text{ simple, has finite measure support}\} \subseteq \bigcap_{p>0} L^p(\mathbb{R}^n)$.
- M is the set of measurable functions on $\mathbb{R}^n$.

Introduction

These notes were written in the summer of 2025 when I taught myself harmonic analysis. I followed Javier Douandikoetxea's *Fourier analysis* and lectures by Joshua Isralowitz at University at Albany on this YouTube channel.

1. Fourier Series and Integrals

1.1. Introduction to Fourier Series and Differentiation.

Definition 1.1. A function $f : \mathbb{R} \to \mathbb{C}$ is 1-periodic if for all $x \in \mathbb{R}$ and $k \in \mathbb{Z}$,

$$f(x+k) = f(x).$$

Assume

$$f(x) = \sum_{k=0}^{\infty} [a_k \cos(2\pi kx) + b_k \sin(2\pi kx)]$$

for some $a_k, b_k \in \mathbb{C}$. Then, equivalently,

$$f(x) = \sum_{k=0}^{\infty} c_{-k} e^{-2\pi ikx} + \sum_{k=1}^{\infty} c_k e^{2\pi ikx}$$

$$= \sum_{k=-\infty}^{\infty} c_k e^{2\pi ikx}$$

where $c_k \in \mathbb{C}$. Multiplying both sides by $e^{-2\pi imx}$ yields

$$e^{-2\pi imx} f(x) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i(k-m)x}.$$

Integrate both sides over $[0,1]$, we get

$$\int_0^1 e^{-2\pi imx} f(x)\, dx = \sum_{k=-\infty}^{\infty} c_k \int_0^1 e^{2\pi i(k-m)x}\, dx$$

$$= \sum_{k=-\infty}^{\infty} c_k \int_0^1 \left[\cos\big(2\pi(k-m)x\big) + i\sin\big(2\pi(k-m)x\big)\right] dx.$$

Notice that

$$\int_0^1 \left[\cos\big(2\pi(k-m)x\big) + i\sin\big(2\pi(k-m)x\big)\right] dx = \begin{cases} 1 & \text{if } k = m, \\ 0 & \text{if } k \neq m. \end{cases}$$

Hence every term in the infinite sum vanishes except the one with $k = m$, yielding

$$c_m = \int_0^1 e^{-2\pi imx} f(x)\, dx.$$

Definition 1.2. For each function $f \in L^1(\mathbb{T})$, the sequence of Fourier coefficients of f is

$$\hat{f}(k) = \int_0^1 e^{-2\pi ikx} f(x)\, dx.$$

Definition 1.3. The Fourier series of f is

$$\sum_{k=-\infty}^{\infty} \hat{f}(k) e^{2\pi ikx}.$$

Proposition 1.4. *If* $\sum_{k=-\infty}^{\infty} |c_k| < \infty$, *and*

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i k x},$$

then $c_k = \hat{f}(k)$.

Question 1.5. Consider, in general, the statement that

$$f(x) \quad \text{``}=\text{''} \quad \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{2\pi i k x}. \tag{1.1}$$

(1) When does 1.1 converge pointwise? uniformly?

(2) If (1) holds, can we approximate f by a trigonometric polynomial? When can we recover f by $\{\hat{f}(k)\}_{k=-\infty}^{\infty}$?

(3) If $f \in L^p_{\text{per}}([0,1])$, $p \geq 1$, does 1.1 converge in $L^p_{\text{per}}([0,1])$?

(4) Does 1.1 converge pointwise a.e. if $f \in L^p_{\text{per}}([0,1])$?

(5) If f is not 1-periodic, can we still define a "Fourier series" somehow?

Remark 1.6. If $f \in L^p_{\text{per}}([0,1])$, then

$$\int_0^1 f(x)\,dx = \int_{a-1}^{a} f(x)\,dx, \quad \forall a \in \mathbb{R}.$$

Definition 1.7. Define the partial sum

$$S_n f(x) = \sum_{k=-n}^{n} \hat{f}(k) e^{2\pi i k x}.$$

Our approach to Question (1) above involves studying $S_n f(x)$ via an associated integral operator. Notice

$$\begin{aligned} S_n f(x) &= \sum_{k=-n}^{n} \left(\int_0^1 e^{-2\pi i k t} f(t) dt \right) e^{2\pi i k x} \\ &= \sum_{k=-n}^{n} \left(\int_0^1 e^{2\pi i k (x-t)} \right) f(t) dt \\ &= \int_0^1 \left(\sum_{k=-n}^{n} e^{2\pi i k (x-t)} \right) f(t) dt, \end{aligned}$$

where

$$D_n(x-t) = \sum_{k=-n}^{n} e^{2\pi i k (x-t)}$$

is the Dirichlet kernel. Let $u = x - t$,

$$\begin{aligned} S_n f(x) &= \int_{x-1}^{x} D_n(u) f(x-u) du \\ &= \int_0^1 D_n(t) f(x-t) dt. \end{aligned}$$

Proposition 1.8. *Properties of the Dirichlet kernel:*

(1) $\int_0^1 D_n(t)\,dt = 1$.

(2) $D_n(t) = \dfrac{\sin\big((2n+1)\pi t\big)}{\sin(\pi t)}$.

Proof. By definition,

$$\int_0^1 D_n(t)\,dt = \sum_{k=-n}^{n} \int_0^1 e^{2\pi ikt}\,dt = 1.$$

For (2),

$$\begin{aligned}
D_n(t) &= \sum_{k=-n}^{n} e^{2\pi ikt} \\
&= e^{-2\pi int} \sum_{k=0}^{2n} e^{2\pi ikt} \\
&= e^{-2\pi int} \sum_{k=0}^{2n} (e^{2\pi it})^k \\
&= e^{-2\pi int} \left(\frac{1 - e^{2\pi i(2n+1)t}}{1 - e^{2\pi it}} \right) \\
&= \frac{e^{-\pi i(2n+1)t}}{e^{-\pi it}} \left(\frac{1 - e^{2\pi i(2n+1)t}}{1 - e^{2\pi it}} \right) \\
&= \frac{e^{-\pi i(2n+1)t} - e^{2\pi i(2n+1)t}}{e^{-\pi it} - e^{\pi it}} \\
&= \frac{\sin\big((2n+1)\pi t \big)}{\sin(\pi t)}.
\end{aligned}$$

□

Notice that

$$\begin{aligned}
f(x) = 1 \cdot f(x) &= \left(\int_0^1 D_n(t)dt \right) f(x) \\
&= \int_0^1 f(x) D_n(t) dt
\end{aligned}$$

looks similar to

$$S_n f(x) = \int_0^1 D_n(t) f(x-t) dt.$$

So we are interested in

$$|f(x) - S_n f(x)| = \left| \int_0^1 \big(f(x) - f(x-t) \big) \frac{\sin\big((2n+1)\pi t \big)}{\sin(\pi t)}\, dt \right|.$$

Proposition 1.9. *Suppose that*

$$\sum_{k=-\infty}^{\infty} |k|^n |c_k| < \infty.$$

If

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi ikx},$$

then

$$f^{(n)}(x) = \sum_{k=-\infty}^{\infty} (2\pi ik)^n c_k e^{2\pi ikx}.$$

Proof. The proof is by induction. Case $n = 1$: Let

$$F(x) = \sum_{k=-\infty}^{\infty} (2\pi ik)c_k e^{2\pi ikx}.$$

Notice

$$\frac{f(x+h)-f(x)}{h} = \sum_{k=-\infty}^{\infty} c_k \left[\frac{e^{2\pi ik(x+h)} - e^{2\pi ikx}}{h}\right]$$
$$= \sum_{k=-\infty}^{\infty} c_k e^{2\pi ikx} \left[\frac{e^{2\pi ikh}-1}{h}\right].$$

Then,

$$\left|\frac{f(x+h)-f(x)}{h} - F(x)\right| = \left|\sum_{k=-\infty}^{\infty} c_k e^{2\pi ikx}\left[\frac{e^{2\pi ikh}-1}{h} - 2\pi ik\right]\right|$$
$$\leq \sum_{k=-\infty}^{\infty} |c_k| \left|\frac{e^{2\pi ikh}-1}{h} - 2\pi ik\right|.$$

By the Mean Value Theorem,

$$\left|\frac{e^{2\pi ikh}-1}{h}\right| \leq \left|\frac{\cos(2\pi kh)-1}{h}\right| + \left|\frac{\sin(2\pi kh)}{h}\right|$$
$$= 2\pi k|\cos(2\pi kh)| + 2\pi k\left|\frac{\sin(2\pi kh)}{2\pi kh}\right|$$
$$\leq 4\pi k.$$

Thus,

$$\left|\frac{e^{2\pi ikh}-1}{h} - 2\pi ik\right| \leq 6\pi k.$$

Let $\varepsilon > 0$, fix $N \in \mathbb{N}$ such that

$$\sum_{|k|>N} (6\pi|k|\,|c_k|) < \frac{\varepsilon}{2}.$$

Then

$$\sum_{k=-\infty}^{\infty} |c_k| \left|\frac{e^{2\pi ikh}-1}{h} - 2\pi ik\right| = \left(\sum_{|k|>N} + \sum_{|k|\leq N}\right) |c_k| \left|\frac{e^{2\pi ikh}-1}{h} - 2\pi ik\right|$$
$$< \sum_{|k|>N} 6\pi|k||c_k| + \sum_{|k|\leq N} |c_k| \left|\frac{e^{2\pi ikh}-1}{h} - 2\pi ik\right|$$
$$< \frac{\varepsilon}{2} + \sum_{|k|\leq N} |c_k| \left|\frac{e^{2\pi ikh}-1}{h} - 2\pi ik\right|.$$

Finally,

$$\lim_{h\to 0}\left|\frac{e^{2\pi ikh}-1}{h} - 2\pi ik\right| = \lim_{h\to 0}\left|\frac{\cos(2\pi kh)-1}{h} + i\Big(\frac{\sin(2\pi kh)}{h} - 2\pi ik\Big)\right| = 0.$$

Pick $\delta > 0$ that $|h| < \delta$ yields

$$\left|\frac{e^{2\pi ikh}-1}{h} - 2\pi ik\right| < \frac{\varepsilon}{2\sum_{|k|\leq N}|c_k|}$$

in which we assume w.l.o.g. $\sum_{|k|\leq N}|c_k| \neq 0$.

Thus $|h| < \delta$ yields

$$\left|\frac{f(x+h)-f(x)}{h} - F(x)\right| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Now we assume the propisition is true for $n-1$, suppose

$$\sum_{k=-\infty}^{\infty} |k|^n |c_k| < \infty,$$

then

$$f^{(n-1)}(x) = \sum_{k=-\infty}^{\infty} (2\pi ik)^{n-1} c_k e^{2\pi ikx}.$$

Let $\widetilde{c}_k = (2\pi ik)^{n-1} c_k$, then

$$f(x) = \sum_{k=-\infty}^{\infty} \widetilde{c}_k e^{2\pi ikx}$$

and clearly,

$$\sum_{k=-\infty}^{\infty} |k||\widetilde{c}_k| < \infty.$$

So,

$$\begin{aligned} f^{(n)}(x) &= \frac{d}{dx} f^{(n-1)}(x) \\ &= \sum_{k=-\infty}^{\infty} (2\pi ik)\widetilde{c}_k e^{2\pi ikx} \\ &= \sum_{k=-\infty}^{\infty} (2\pi ik)^n e^{2\pi ikx}. \end{aligned}$$

□

Proposition 1.10. *If $f \in C^n([0,1])$ and*

$$f^{(\ell)}(0) = f^{(\ell)}(1)$$

for $l = 0, 1, \dots, n-1$, then

$$\hat{f}(k) = (2\pi ik)^{-n} \widehat{f^{(n)}}(k).$$

Proof. The proof follows from induction and inetgration by parts. When $n = 1$,

$$\begin{aligned} \hat{f}(k) &= \int_0^1 e^{-2\pi ikx} f(x)\, dx \\ &= \frac{e^{-2\pi ikx}}{-2\pi ik} f(x)\Big|_0^1 - \int_0^1 \frac{e^{-2\pi ikx}}{-2\pi ik} f'(x)\, dx \\ &= \frac{1}{2\pi ik} \int_0^1 e^{-2\pi ikx} f'(x)\, dx. \end{aligned}$$

Assume true for $n-1$, we have

$$\begin{aligned} \hat{f}(k) &= (2\pi ik)^{-n+1} \int_0^1 e^{-2\pi ikx} f^{(n-1)}(x)\, dx \\ &= (2\pi ik)^{-n+1} \left[\frac{e^{-2\pi ikx}}{-2\pi ik} f^{(n-1)}(x)\Big|_0^1 - \int_0^1 \frac{e^{-2\pi ikx}}{-2\pi ik} f^{(n)}(x)\, dx \right] \\ &= (2\pi ik)^{-n} \widehat{f^{(n)}}(k). \end{aligned}$$

□

Corollary 1.11. *If $f(x)$ is as in the previous proposition, if*

$$f(x) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{2\pi i k x}$$

and if

$$\sum_{k=-\infty}^{\infty} |\widehat{f^{(n)}}(k)| < \infty,$$

then

$$f^{(n)}(x) = \sum_{k=-\infty}^{\infty} (2\pi i k)^n \hat{f}(k) e^{2\pi i k x}.$$

Proof. The proof follows from the previous proposition that

$$\sum_{|k|=0}^{\infty} |k|^n |\hat{f}(k)|$$

$$= \sum_{|k|=0}^{\infty} |k|^n |(2\pi i k)^{-n} \widehat{f^{(n)}}(k)| < \infty.$$

□

Remark 1.12 (Riemann-Lebesgue Lemma)**.** If $f \in L^1([0,1])$, then $\lim_{|k|\to\infty} |\hat{f}(k)| = 0$.

Corollary 1.13. *If f is as in the previous proposition, then the Fourier series of $f^{(n)}(x)$ is obtained by n termwise differentiations of the Fourier series of $f(x)$.*

Proof. The Fourier series of $f^{(n)}$ is

$$\sum_{k=-\infty}^{\infty} \widehat{f^{(n)}}(k) e^{2\pi i k x} = \sum_{k=-\infty}^{\infty} (2\pi i k)^n \hat{f}(k) e^{2\pi i k x}$$

$$= \sum_{k=-\infty}^{\infty} \frac{d^n}{dx^n} [\hat{f}(k) e^{2\pi i k x}].$$

□

1.2. **Convergence of Fourier Series.**

Example 1.14. Extending $f(x) = x(1-x)$ as a periodic function yields

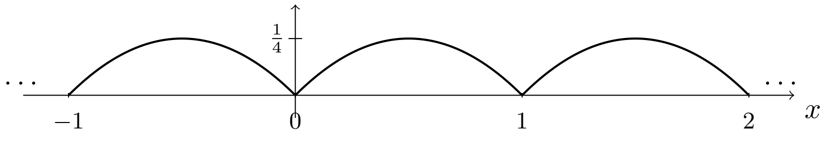

FIGURE 1. *"Periodic"* $x(1-x)$

Now we compute its Fourier coefficient

$$\hat{f}(k) = \int_0^1 e^{-2\pi i k x} x(1-x) dx$$

$$= \frac{-1}{2\pi^2 k^2}.$$

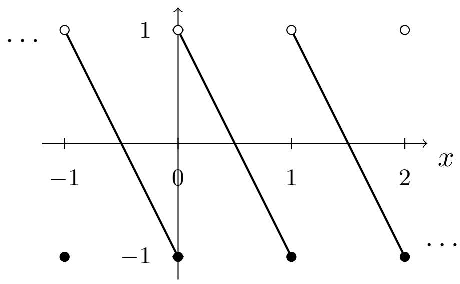

FIGURE 2. $f'(x)$

The Fourier series of $f(x)$ is

$$f(x) = \frac{1}{6} - \frac{1}{2\pi^2}(e^{2\pi ix} + e^{-2\pi ix}) - \frac{1}{8\pi^2}(e^{4\pi ix} + e^{-4\pi ix})$$
$$= \frac{1}{6} - \frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos(2\pi kx)}{k^2}.$$

Lemma 1.15 (Summation by parts)**.** *If $S_n = \sum_{k=1}^{n} a_k$, $n \geq 2$, then*

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n-1} S_k(b_k - b_{k+1}) + S_n b_n.$$

Proof. The proof is by induction. □

Theorem 1.16 (Dirichlet's Test)**.** *Suppose the following*

(1) $\{b_k\}_{k=1}^{\infty}$ *is monotonic,*

(2) $\lim_{k\to\infty} b_k = 0$,

(3) $|\sum_{k=1}^{n} a_k(x)| \leq M \quad \forall x \in A, n \in \mathbb{N}$.

Then $\sum_{k=1}^{\infty} b_k a_k(x)$ converges uniformly on A.

Proof. Let $\varepsilon > 0$, we find $N \in \mathbb{N}$ such that for $n > m > N$, we have

$$\left\| \sum_{k=1}^{n} b_k a_k(x) - \sum_{k=1}^{m} b_k a_k(x) \right\|_{\infty, A} < \varepsilon.$$

Notice

$$\left| \sum_{k=1}^{n} b_k a_k(x) - \sum_{k=1}^{m} b_k a_k(x) \right|$$
$$= \left| \left[\sum_{k=1}^{n-1} S_k(x)(b_k - b_{k+1}) + S_n(x) b_n \right] - \left[\sum_{k=1}^{m-1} S_k(x)(b_k - b_{k+1}) + S_m(x) b_m \right] \right|$$
$$= \left| \sum_{k=m}^{n-1} S_k(x)(b_k - b_{k+1}) + S_n(x) b_n - S_m(x) b_m \right|$$
$$\leq \sum_{k=m}^{n-1} |S_k(x)||b_k - b_{k+1}| + |S_n(x)||b_n| + |S_m(x)||b_m|$$

Since $|\sum_{k=1}^{n} a_k(x)| \leq M$, and assume w.l.o.g. $\{b_k\}$ is increasing,

$$\leq \sum_{k=m}^{n-1} M|b_{k+1} - b_k| + M(|b_n| + |b_m|)$$

$$= M\sum_{k=m}^{n-1}(b_{k+1}-b_k) + M(|b_n|+|b_m|)$$
$$= M(b_n-b_m) + M(|b_n|+|b_m|)$$
$$\leq 2M(|b_n|+|b_m|).$$

We want this $< \varepsilon$, so pick $N \in \mathbb{N}$ such that for $m > N$, we have

$$|b_m| < \frac{\varepsilon}{4M}.$$

□

Lemma 1.17. *If $x \in \mathbb{Z}$, then*

(i)

$$\sum_{k=0}^{n-1}\sin(2\pi kx) = \frac{\sin(n\pi x)\sin((n-1)\pi x)}{\sin(\pi x)}$$

(ii)

$$\sum_{k=0}^{n-1}\cos(2\pi kx) = \frac{\sin(n\pi x)\cos((n-1)\pi x)}{\sin(\pi x)}$$

Proposition 1.18. $\sum_{k=1}^{\infty}\frac{\sin(2\pi kx)}{k}$ *converges pointwise on $\mathbb{R}$ and uniformly on $A \subseteq \mathbb{R}$ with $d(A,\mathbb{Z}) = \inf\{|x-n| : x \in A, n \in \mathbb{Z}\} > 0$, i.e. away from integers.*

Proof. Let $b_k = \frac{1}{k}$, $a_k(x) = \sin(2\pi kx)$, then

$$\left|\sum_{k=0}^{n} a_k(x)\right| = \frac{|\sin((n+1)\pi x)||\sin(n\pi x)|}{|\sin(\pi x)|} \leq \frac{1}{C}$$

for some $C = C(A) > 0$. By Dirichlet's test,

$$\sum_{k=1}^{\infty}\frac{\sin(2\pi kx)}{k}$$

converges uniformly on A. Note that

$$\sum_{k=1}^{\infty}\frac{\sin(2\pi km)}{k} = 0 \text{ when } m \in \mathbb{Z}.$$

This is what typically happens to the Fourier series when we have jumps as in 1.14.

Non-uniform convergence: Let $F, F_n : \mathbb{R} \to \mathbb{R}$ such that

$$F(x) = \sum_{k=1}^{\infty}\frac{\sin(2\pi kx)}{k}, \quad F_n(x) = \sum_{k=1}^{n}\frac{\sin(2\pi kx)}{k}.$$

We claim that $F_n \not\to F$ uniformly on $[0,a]$ for any $a > 0$. It can be proved as follows. Let $x_n = \frac{1}{8n}$, then

$$F_n(x_n) = \sum_{k=1}^{n}\frac{\sin\left(\frac{\pi k}{4n}\right)}{k}.$$

By the mean value theorem,

$$F_n(x_n) \geq \sum_{k=1}^{n}\frac{\sqrt{2}}{2}\left(\frac{\pi k}{4n}\right)\cdot\frac{1}{k}$$
$$= \frac{\sqrt{2}\pi}{8}.$$

If $F_n \to F$ uniformly on $[0, a]$, then F is continuous on $[0, a]$. Since $F(0) = 0$, fix n where $|F(x_n)| < \frac{\sqrt{2}\pi}{16}$, then

$$|F(x_n) - F_n(x_n)| \leq \sup_{x \in [0,a]} |F(x) - F_n(x)| < \frac{\sqrt{2}\pi}{16}.$$

Then we have the contradiction that

$$\begin{aligned} |F_n(x_n)| &\leq |F_n(x_n) - F(x_n)| + |F(x_n)| \\ &< \frac{\sqrt{2}\pi}{8}. \end{aligned}$$

□

Recall that

$$S_n f(x) = \sum_{k=-n}^{n} \hat{f}(k) e^{2\pi i k x},$$

and

$$|f(x) - S_n f(x)| = \left| \int_0^1 \big(f(x+t) - f(x) \big) \frac{\sin\big(\pi(2n+1)t\big)}{\sin(\pi t)} \, dt \right|.$$

Fix x and let $G_x(t) = \frac{f(x+t)-f(x)}{\sin \pi t}$. Notice that the function $\sin\big(\pi(2n+1)t\big)$ has period $\frac{3}{2n+1}$ as shown in the figure below. As n grows larger, we have more periods in the interval $[0, 1]$.

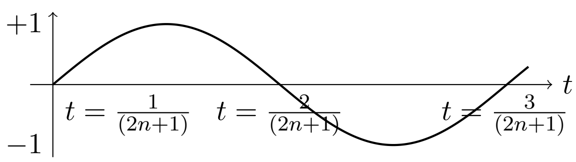

FIGURE 3. $\sin\big(\pi(2n+1)t\big)$

The rough idea is to let $t_k = \frac{2k}{2n+1}$, we can write

$$\begin{aligned} [0,1] &= \left(\bigcup_{k=0}^{n-1} \left[\frac{2k}{2n+1}, \frac{2k+1}{2n+1} \right) \right) \cup \left[\frac{2n}{2n+1}, 1 \right] \\ &= \left(\bigcup_{k=0}^{n-1} [t_k, t_{k+1}) \right) \cup [t_n, 1]. \end{aligned}$$

If we try to approximate by taking $G_x(t) \approx G_x(t_k)$ on $[t_k, t_{k+1}]$, and $G_x(t) \approx G_x(t_n)$ on $[t_n, 1]$, then

$$\begin{aligned} |f(x) - S_n f(x)| &\leq \sum_{k=0}^{n-1} \left| \int_{t_k}^{t_{k+1}} G_x(t) \sin\big(\pi(2n+1)t\big) dt \right| + \left| \int_{t_n}^{1} G_x(t) \sin\big(\pi(2n+1)t\big) dt \right| \\ &\approx \sum_{k=0}^{n-1} |G_x(t_k)| \left| \int_{t_k}^{t_{k+1}} \sin\big(\pi(2n+1)t\big) dt \right| + |G_x(t_n)| \left| \int_{t_n}^{1} \sin\big(\pi(2n+1)t\big) dt \right| \\ &\approx 0 \end{aligned}$$

by the fundamental theorem of calculus.

Remark 1.19. We can also use this approach to prove the Riemann-Lebesgue lemma.

Another sketch of proof of the Riemann-Lebesgue Lemma.

$$\hat{f}(n) = \int_0^1 f(t)e^{-2\pi int}dt,$$
$$\leq \left|\int_0^1 f(t)\sin(2\pi nt)dt\right| + \left|\int_0^1 f(t)\cos(2\pi nt)dt\right|$$

Notice that

$$\left|\int_0^1 f(t)\sin(2\pi nt)dt\right| \leq \sum_{k=0}^{n-1}\left|\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\sin(2\pi nt)dt\right|$$
$$\approx \sum_{k=0}^{n-1}|f(\tfrac{k}{n})|\left|\int_{\frac{k}{n}}^{\frac{k+1}{n}} \sin(2\pi nt)dt\right| \approx 0.$$

Similarly treatment of the $\cos(2\pi nt)$ term proves the lemma. □

Theorem 1.20 (Dini's Criterion)**.** *Let $f \in L^1_{per}([0,1])$ be defined at $x \in [0,1]$. Suppose $\exists\, \delta > 0$ such that*

$$\int_{-\delta}^{\delta}\left|\frac{f(x+t)-f(x)}{t}\right| dt < \infty,$$

then

$$\lim_{n\to\infty} S_n f(x) = f(x).$$

Proof. Notice

$$|S_n f(x) - f(x)| = \left|\int_0^1 \big(\, f(x+t) - f(x)\,\big)\frac{\sin\big(\,\pi(2n+1)t\,\big)}{\sin \pi t}dt\right|$$
$$\leq \left|\underbrace{\left(\int_0^{\delta} + \int_{\delta}^{1}\right)\big(\, f(x+t) - f(x)\,\big)\frac{\sin\big(\,\pi(2n+1)t\,\big)}{\sin \pi t}dt}_{A}\right|$$
$$+ \left|\underbrace{\int_{\delta}^{1-\delta}\big(\, f(x+t) - f(x)\,\big)\frac{\sin\big(\,\pi(2n+1)t\,\big)}{\sin \pi t}dt}_{B}\right|$$

where

$$A = \left(\int_0^{\delta} + \int_{\delta}^{1}\right) G_x(t)\sin\big(\,\pi(2n+1)t\,\big)dt$$
$$= \left(\int_0^{\delta} + \int_0^{1}\right)\frac{G_x(t)}{2i}e^{i\pi(2n+1)t}dt - \left(\int_0^{\delta} + \int_0^{1}\right)\frac{G_x(t)}{2i}e^{-i\pi(2n+1)t}dt$$
$$= (\widehat{\frac{G_x}{2i}\chi_{[0,\delta]}})(2n+1) + (\widehat{\frac{G_x}{2i}\chi_{[\delta,1]}})(2n+1)$$
$$-(\widehat{\frac{G_x}{2i}\chi_{[0,\delta]}})(-2n-1) - (\widehat{\frac{G_x}{2i}\chi_{[\delta,1]}})(-2n-1)$$

which $\to 0$ as $n \to \infty$ by the Riemann-Lebesgue lemma. The L^1 requirement of the can be checked through the following.

$$\int_0^1\left|\frac{G_x(t)}{2i}\chi_{[0,\delta]}(t)\right| dt = \frac{1}{2}\int_0^{\delta}\left|\frac{f(x+t)-f(x)}{\sin \pi t}\right| dt$$
$$= \frac{1}{2}\int_0^{\delta}\left|\frac{f(x+t)-f(x)}{t}\right|\left|\frac{t}{\sin \pi t}\right| dt$$

$$\leq \frac{1}{2}\frac{\delta}{\sin(\pi\delta)}\int_0^\delta \left|\frac{f(x+t)-f(x)}{t}\right| dt < \infty,$$

and

$$\int_0^1 \left|\frac{G_x(t)}{2i}\chi_{[\delta,1]}(t)\right| dt = \int_\delta^1 \left|\frac{f(x+t)-f(x)}{\sin \pi t}\right| dt$$
$$= \int_{-\delta}^0 \left|\frac{f(x+t)-f(x)}{\sin \pi t}\right| dt < \infty.$$

Similarly, $B \to 0$ by the Riemann-Lebesgue lemma, since

$$\int_\delta^{1-\delta} \left|\frac{f(x+t)-f(x)}{\sin \pi t}\right| dt \leq \frac{1}{\sin(\pi\delta)}\int_0^1 |f(x+t)-f(x)|dt$$
$$\leq \frac{\|f\|_{L^1([0,1])}+|f(x)|}{\sin(\pi\delta)} < \infty.$$

□

Example 1.21. Examples of functions satisfying Dini condition:

(1) Hölder continuity near x: For some $\alpha \in (0,1]$, there exists $\delta > 0$ such that
$$|f(x+t)-f(x)| < C|t|^\alpha, \quad \text{for } |t| < \delta.$$

(2) f is right and left differentiable at x.

(3) Let
$$f(t) = \begin{cases} 1 & \text{if } t \in \mathbb{Q}\setminus\mathbb{Z} \\ 0 & \text{if } t \in (\mathbb{R}\setminus\mathbb{Q})\cup\mathbb{Z}, \end{cases}$$
i.e. $f(0)=0$, $f(t)=0$ a.e.

Remark 1.22. Dini condition $\not\Rightarrow$ continuity, and continuity $\not\Rightarrow$ Dini condition.

Example 1.23. We can construct a periodic continuous function using

$$f(t) = \begin{cases} 0 & \text{if } t = 0, \\ \frac{1}{\ln(|t|)} & \text{if } 0 < |t| < 1. \end{cases}$$

Then,

$$\int_0^\delta \left|\frac{f(t+0)-f(0)}{t}\right| dt = -\int_0^\delta \frac{1}{t\ln(t)}\,dt = \infty$$

Definition 1.24. Define

$$C_{\text{per}}(\mathbb{R}) = \left\{f : \mathbb{R} \to \mathbb{C} \text{ continuous, 1-periodic, } \|f\|_{C_{\text{per}}} < \infty\right\}$$

with the norm

$$\|f\|_{C_{\text{per}}(\mathbb{R})} = \max_{t\in[0,1]} |f(t)|.$$

Remark 1.25. Fix $X = \{x_j\}_{j=1}^\infty \subseteq [0,1]$. Let

$$D = \{f \in C_{\text{per}}(\mathbb{R}) : \{S_n f(t)\}_{n=1}^\infty \text{ diverges for every } t \in X\}.$$

We claim that D is dense in $C_{\text{per}}([0,1])$. To prove the claim, we need Baire Category Theorem.

Theorem 1.26 (Baire Category Theorem)**.** *If M is a complete metric space, then the countable intersection of open dense subsets is dense.*

Definition 1.27. Recall: If M, N are normed vector spaces and $T : M \to N$, then the operator norm of T is

$$\|T\|_{\text{op}} = \sup_{\|x\|_M=1} \|T(x)\|_N.$$

Lemma 1.28. *Let M be a Banach space. Let $\{T_i\}_{i=1}^{\infty} : M \to N$ be a countable family of bounded linear maps with*

$$\sup_{i\in\mathbb{N}} \|T_i\|_{op} = +\infty.$$

Then,

$$R = \left\{ x \in M : \sup_{i\in\mathbb{N}} \|T_i(x)\|_N = +\infty \right\}$$

is the countable intersection of open, dense subsets of M.

Proof. Clearly,

$$R = \bigcap_{m=1}^{\infty} \left\{ x \in M : \sup_{i\in\mathbb{N}} \|T_i(x)\|_N > m \right\} = \bigcap_{m=1}^{\infty} R_m$$

Claim: Each R_m is open and dense in M.

Openness: Homework.

Density: Let $x \in R_m^c$, so

$$\sup_{i\in\mathbb{N}} \|T_i(x)\|_N \leq m.$$

Let $\varepsilon > 0$. Since $\sup_{i\in\mathbb{N}} \|T_i\|_{\text{op}} = \infty$, pick i such that

$$\|T_i\|_{\text{op}} = \sup_{\|x\|_M = 1} \|T_{i'}(x)\|_N > \frac{4m}{\varepsilon}.$$

Pick $x' \in M$ with $\|x'\|_M = 1$ such that $\|T_{i'}(x')\|_N > \frac{4m}{\varepsilon}$. Let

$$x'' = x + \frac{\varepsilon}{2}x'.$$

Then,

$$\begin{aligned} \|x - x''\|_M &= \left\| x - \left(x + \frac{\varepsilon}{2}x' \right) \right\|_M \\ &= \frac{\varepsilon}{2}\|x'\|_M \\ &= \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

Notice

$$\begin{aligned} \|T_{i'}(x'')\|_N &= \left\| T_{i'}\left(\frac{\varepsilon}{2}x' \right) - T_{i'}(-x) \right\|_N \\ &\geq \frac{\varepsilon}{2}\|T_{i'}(x')\|_N - \|T_{i'}(x)\|_N \\ &> \frac{\varepsilon}{2}\left(\frac{4m}{\varepsilon} \right) - m = m. \end{aligned}$$

Hence $x'' \in R_m$, R_m is dense. □

Proposition 1.29. *Fix $x \in [0,1]$. Let $T_{n,x} : C_{per}(\mathbb{R}) \to \mathbb{C}$ be defined as*

$$T_{n,x}f = S_n f(x).$$

Then, for $\forall x \in [0,1]$,

$$\sup_{n\in\mathbb{N}} \|T_{n,x}\|_{op} = \infty.$$

Proof. Step 1: Notice that

$$\int_0^1 \left|\frac{\sin\big(\pi(2n+1)t\big)}{\sin(\pi t)}\right| dt \geq \int_0^1 \left|\frac{\sin\big(\pi(2n+1)t\big)}{\pi t}\right| dt$$

Change variable $u = (2n+1)t$, then

$$= \int_0^{2n+1} \left|\frac{\sin(\pi u)}{\pi u}\right| du \geq \sum_{k=0}^{n} \int_k^{k+1} \left|\frac{\sin(\pi u)}{\pi u}\right| du$$

$$\geq \frac{1}{\pi}\sum_{k=0}^{n} \frac{1}{k+1}\int_k^{k+1} |\sin(\pi u)| du$$

$$= \frac{2}{\pi}\sum_{k=0}^{n} \frac{1}{k+1}$$

which diverges as $n \to \infty$.

Step 2: Assume there exists $\{f_j\}_{j=1}^{\infty} \subseteq C_{\text{per}}(\mathbb{R})$ with $\|f_j\|_{C_{per}(\mathbb{R})} = 1$, such that

$$\lim_{j\to\infty} f_j(x-t)\frac{\sin\big(\pi(2n+1)t\big)}{\sin(\pi t)} = \left|\frac{\sin\big(\pi(2n+1)t\big)}{\sin(\pi t)}\right|$$

pointwise for $t \neq \frac{k}{2n+1}$, $k = 1, \dots, 2n$. Then,

$$\|T_{n,x}\|_{\text{op}} \geq \lim_{j\to\infty} |T_{n,x} f_j|$$

$$= \lim_{j\to\infty} \int_0^1 f_j(x-t)\frac{\sin\big(\pi(2n+1)t\big)}{\sin(\pi t)} dt$$

$$= \int_0^1 \left|\frac{\sin\big(\pi(2n+1)t\big)}{\sin(\pi t)}\right| dt$$

$$\geq \frac{2}{\pi}\sum_{k=0}^{n} \frac{1}{k+1}.$$

So

$$\sup_{n\in\mathbb{N}} \|T_{n,x}\|_{\text{op}} = \infty.$$

Step 3: We now prove the assumption used in step 2.

Let

$$\widetilde{f_j}(t) = \operatorname{sgn}\left(\frac{\sin(\pi\big(2n+1)t\big)}{\sin(\pi t)}\right)$$

$$= \left(\frac{\sin(\pi\big(2n+1)t\big)}{\sin(\pi t)}\right) \Bigg/ \left|\frac{\sin(\pi\big(2n+1)t\big)}{\sin(\pi t)}\right| \in \{\pm 1\}$$

for $t \in I_j$ where

$$I_j = \left[0, \frac{1}{2n+1} - \frac{1}{j}\right) \cup \left(\bigcup_{k=1}^{2n-1} \left[\frac{1}{j} + \frac{k}{2n+1}, \frac{k+1}{2n+1} - \frac{1}{j}\right)\right)$$

are the intervals avoiding points $t = \frac{k}{2n+1}$, and interpolate between them to define $\tilde{f}_j$ as piecewise linear, i.e. for $t \notin I_j$,

$$\widetilde{f_j}(t) = \text{line segment connecting } \widetilde{f_j}\left(\frac{k}{2n+1} - \frac{1}{j}\right) \text{ and } \widetilde{f_j}\left(\frac{1}{j} + \frac{k}{2n+1}\right).$$

So for $t \neq \frac{1}{2n+1}, \dots, \frac{2n}{2n+1}$,

$$\lim_{j\to\infty} \widetilde{f}_j(t) = \operatorname{sgn}\left(\frac{\sin(\pi\big(2n+1)t\big)}{\sin(\pi t)}\right).$$

Then set

$$f_j(t) = \widetilde{f}_j(x-t).$$

So

$$\begin{aligned} f_j(x-t) &= \widetilde{f}_j\big(x-(x-t)\big) \\ &= \widetilde{f}_j(t). \end{aligned}$$

Thus

$$\begin{aligned} &\lim_{j\to\infty} f_j(x-t)\frac{\sin\big(\pi(2n+1)t\big)}{\sin(\pi t)} \\ &= \operatorname{sgn}\left(\frac{\sin(\pi\big(2n+1)t\big)}{\sin(\pi t)}\right)\frac{\sin\big(\pi(2n+1)t\big)}{\sin(\pi t)} \\ &= \left|\frac{\sin\big(\pi(2n+1)t\big)}{\sin(\pi t)}\right| \end{aligned}$$

for $t \neq \frac{1}{2n+1}, \dots, \frac{2n}{2n+1}$. □

We now proceed to prove our previous claim.

Theorem 1.30. *Fix $X = \{x_j\}_{j=1}^{\infty} \subseteq [0,1]$. Let*

$$D = \{f \in C_{per}(\mathbb{R}) : \{S_n f(t)\}_{n=1}^{\infty} \text{ diverges for every } t \in X\}.$$

Then D is dense in $C_{per}([0,1])$.

Proof. Note that

$$\begin{aligned} |S_n f(x)| &\leq \int_0^1 |f(x+t)| \left|\frac{\sin\big(\pi(2n+1)t\big)}{\sin(\pi t)}\right| dt \\ &\leq \|f\|_{C_{\text{per}}(\mathbb{R})}(2n+1). \end{aligned}$$

Hence,

$$\|T_{n,x}\|_{\text{op}} \leq 2n+1.$$

Since for all $x \in [0,1]$, $\sup_{n\in\mathbb{N}} \|T_{n,x}\|_{\text{op}} = \infty$,

$$R_x = \left\{f \in C_{\text{per}}(\mathbb{R}) : \sup_{n\in\mathbb{N}} |T_{n,x} f| = \infty\right\}$$

is the countable intersection of open dense subsets of $C_{\text{per}}(\mathbb{R})$. Finally, write

$$R_{x_j} = \bigcap_{m=1}^{\infty} (R_{x_j})_m,$$

as an countable intersection of open dense subsets of $C_{\text{per}}(\mathbb{R})$. Then,

$$\bigcap_{j=1}^{\infty} R_{x_j} = \bigcap_{j=1}^{\infty}\bigcap_{m=1}^{\infty} (R_{x_j})_m$$

is dense in $C_{\text{per}}(\mathbb{R})$ by the Baire category theorem. □

To prove Jordan's Theorem, we introduce the following key tool:

Lemma 1.31 (Bonnet's Mean Value Theorem). *Let $f \in C([a,b])$ and $g \geq 0$ be increasing on $[a,b]$. Then there exists $c \in (a,b)$ such that*

$$\int_a^b f(x)g(x)\,dx = g(b)\int_a^b f(x)\,dx.$$

Theorem 1.32 (Jordan's Theorem). *Let $x \in [0,1]$ and assume $f \in L^1_{per}(0,1)$ is monotone near x. Then*

$$\lim_{n\to\infty} S_n f(x) = \frac{f(x+)+f(x-)}{2},$$

where

$$f(x+) = \lim_{t\to x^+} f(t), \quad f(x-) = \lim_{t\to x^-} f(t).$$

Proof. Note that

$$S_n f(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x-t)\frac{\sin\big(\pi(2n+1)t\big)}{\sin(\pi t)}dt.$$

Let $u=-t$, $du=-dt$,

$$S_n f(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x+u)\frac{\sin\big(\pi(2n+1)u\big)}{\sin(\pi u)}du.$$

Summing the above together yields

$$S_n f(x) = \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} [f(x+t)+f(x-t)]\,\frac{\sin\big(\pi(2n+1)t\big)}{\sin(\pi t)}dt.$$

Since the integrand is even,

$$= \int_0^{\frac{1}{2}} [f(x+t)+f(x-t)]\,\frac{\sin\big(\pi(2n+1)t\big)}{\sin(\pi t)}dt.$$

Then it's enough to prove:

(A)

$$\lim_{n\to\infty}\int_0^{\frac{1}{2}} f(x+t)\frac{\sin\big(\pi(2n+1)t\big)}{\sin(\pi t)}dt = \frac{f(x+)}{2},$$

(B)

$$\lim_{n\to\infty}\int_0^{\frac{1}{2}} f(x-t)\frac{\sin\big(\pi(2n+1)t\big)}{\sin(\pi t)}dt = \frac{f(x-)}{2}.$$

(A): Let $g(t) = f(x+t)$. So need to show

$$\lim_{n\to\infty}\int_0^{\frac{1}{2}} g(t)\frac{\sin\big(\pi(2n+1)t\big)}{\sin(\pi t)}dt = \frac{g(0+)}{2}.$$

Recall that

$$\int_0^{\frac{1}{2}} \frac{\sin\big(\pi(2n+1)t\big)}{\sin(\pi t)}dt = \frac{1}{2}\int_0^1 \frac{\sin\big(\pi(2n+1)t\big)}{\sin(\pi t)}dt = \frac{1}{2}.$$

So

$$\frac{g(0+)}{2} = \int_0^{\frac{1}{2}} g(0+)\frac{\sin\big(\pi(2n+1)t\big)}{\sin(\pi t)}dt.$$

Hence, it suffices to show that

$$\lim_{n\to\infty}\underbrace{\int_0^{\frac{1}{2}} \big(g(t)-g(0+)\big)\frac{\sin\big(\pi(2n+1)t\big)}{\sin(\pi t)}dt}_{(*)} = 0.$$

Let $\varepsilon > 0$, pick $\delta > 0$ such that g increasing on $[0, \delta]$, and $|g(t) - g(0+)| < \varepsilon$ if $0 < t \leq \delta$. We can write $(*)$ as

$$\int_0^{\frac{1}{2}} = \int_0^{\delta} + \int_{\delta}^{\frac{1}{2}}.$$

First integral: Let

$$\widetilde{g}(t) = \begin{cases} 0 & t = 0, \\ g(t) - g(0+) & t \in (0, \delta]. \end{cases}$$

By Bonnet's Mean Value Theorem, there exists $c = c(n, \delta) \in (0, \delta)$ such that

$$\begin{aligned} &\left| \int_0^{\delta} \widetilde{g}(t) \frac{\sin\big(\pi(2n+1)t\big)}{\sin(\pi t)} dt \right| \\ &\leq |\widetilde{g}(\delta)| \left| \int_c^{\delta} \frac{\sin\big(\pi(2n+1)t\big)}{\sin(\pi t)} dt \right| \\ &< \varepsilon \left| \int_c^{\delta} \frac{\sin\big(\pi(2n+1)t\big)}{\sin(\pi t)} dt \right|. \end{aligned}$$

If we take the supremum over all c and δ, the integral is still bounded, so there exists a constant C such that

$$\left| \int_0^{\delta} \widetilde{g}(t) \frac{\sin\big(\pi(2n+1)t\big)}{\sin(\pi t)} dt \right| < C\varepsilon.$$

Second integral:

$$\int_0^1 \big(g(t) - g(0+)\big) \chi_{[\delta, \frac{1}{2}]} \frac{\sin\big(\pi(2n+1)t\big)}{\sin(\pi t)} dt \to 0 \quad \text{as } n \to \infty$$

by the Riemann-Lebesgue lemma (similar to proof of Dini's Criterion).

(B): Similar to (A), homework. □

Definition 1.33. Let $f : (a, b) \to \mathbb{C}$, $x \in (a, b)$. Define

$$f'(x+) = \lim_{s \to x^+} \frac{f(s) - f(x+)}{s - x},$$

$$f'(x-) = \lim_{s \to x^-} \frac{f(s) - f(x-)}{s - x}.$$

Theorem 1.34 (Dirichlet's Theorem)**.** *Let $f : [0, 1] \to \mathbb{C}$, $f \in L^1_{per}([0, 1])$, and $x \in [0, 1]$. Suppose $f'(x+)$ and $f'(x-)$ exist. Then*

$$\lim_{n \to \infty} S_n f(x) = \frac{f(x+) + f(x-)}{2}.$$

Proof. Homework! □

Example 1.35. Construct a 1-periodic function from

$$f(x) = \begin{cases} 1 - 2x & \text{on } (0, 1), \\ 0 & \text{if } x = 0, 1. \end{cases}$$

Then clearly, $f(0+) = -1 = f(0-)$ and

$$\frac{f(0+) + f(0-)}{2} = 0.$$

Hence the Fourier series $\frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2k\pi x)}{k}$ converges to 0 when $x = 0$ by Jordan's theorem (which doesn't really tell us much since $\sin 0 = 0$).

Example 1.36. Construct a 1-periodic function from

$$f(x) = x(1-x).$$

Then $f'(0+) = 1, \quad f'(0-) = -1$, and

$$\frac{f(0+) + f(0-)}{2} = 0.$$

Hence, its Fourier series

$$\frac{1}{6} - \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{\cos(2k\pi x)}{k^2}.$$

converges to 0 when $x = 0$, yielding a surprising result:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

1.3. **Summability methods.** General philosophy: average!

Example 1.37. If $\lim_{n\to\infty} x_n = x$, then

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} x_n = x.$$

Proof. We have

$$\begin{aligned}\left|\left(\frac{1}{N}\sum_{n=0}^{N-1} x_n\right) - x\right| &= \frac{1}{N}\sum_{n=0}^{N-1} |x_n - x| \\ &\le \frac{1}{N}\sum_{n=0}^{N-1} |x_n - x|.\end{aligned}$$

Let $\varepsilon > 0$, pick M such that for $n > M$,

$$|x_n - x| < \frac{\varepsilon}{2}.$$

Then, for some constant C,

$$\begin{aligned}\frac{1}{N}\sum_{n=0}^{N-1} |x_n - x| &= \frac{1}{N}\sum_{n=0}^{M} |x_n - x| + \frac{1}{N}\sum_{n=M+1}^{N-1} |x_n - x| \\ &\le \frac{C(M+1)}{N} + \frac{1}{N}\sum_{n=M+1}^{N-1} \frac{\varepsilon}{2} \\ &= \frac{C(M+1)}{N} + \frac{\varepsilon}{2N}\sum_{n=M+1}^{N-1} 1 \\ &\le \frac{C(M+1)}{N} + \frac{\varepsilon}{2} < \varepsilon\end{aligned}$$

for $N > \max\left\{M+1, \frac{\varepsilon}{2CM}\right\}$. □

Remark 1.38. The converse is false. A counterexample could be $x_n = (-1)^{n+1}$, then

$$\sum_{n=0}^{N-1} x_n = \begin{cases} -1 & \text{if } N \text{ is even,} \\ 0 & \text{if } N \text{ is odd.}\end{cases}$$

Then, $\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{\infty} x_n = 0$, while $\lim_{n\to\infty} x_n$ does not exist.

Definition 1.39. Define

$$\sigma_N(f)(x) = \frac{1}{N}\sum_{n=0}^{N-1} S_n(f)(x)$$

$$= \frac{1}{N}\sum_{n=0}^{N-1}\left[\int_0^1 f(x-t)\frac{\sin\big(\pi(2n+1)t\big)}{\sin(\pi t)}dt\right]$$

$$= \int_0^1 f(x-t)\left[\frac{1}{N}\sum_{n=0}^{N-1}\frac{\sin\big(\pi(2n+1)t\big)}{\sin(\pi t)}\right]dt$$

$$= \int_0^1 f(x-t)F_N(t)dt.$$

where the Fejér kernel is

$$F_N(t) = \frac{1}{N}\sum_{n=0}^{N-1}\frac{\sin\big(\pi(2n+1)t\big)}{\sin(\pi t)}.$$

Remark 1.40. Notice that

$$\sum_{n=0}^{N-1}\sin\big(\pi(2n+1)t\big) = \operatorname{Im}\sum_{n=0}^{N-1} e^{i\pi(2n+1)t}$$

$$= \operatorname{Im}\left[e^{i\pi t}\sum_{n=0}^{N-1}(e^{2\pi it})^n\right]$$

$$= \operatorname{Im}\left[e^{i\pi t}\left(\frac{1-e^{2\pi iNt}}{1-e^{2\pi it}}\right)\right]$$

$$= \operatorname{Im}\left[e^{-\pi iNt}\left(\frac{e^{-\pi iNt}-e^{\pi iNt}}{e^{-i\pi t}-e^{i\pi t}}\right)\right]$$

$$= \frac{\sin(\pi Nt)\sin(\pi Nt)}{\sin(\pi t)}.$$

So,

$$F_N(t) = \frac{1}{N}\left(\frac{\sin(N\pi t)}{\sin(\pi t)}\right)^2 \geq 0.$$

$F_n(t)$ has the following key properties:

(1)

$$\int_0^1 |F_N(t)|dt = \int_0^1 F_N(t)dt$$

$$= \frac{1}{N}\sum_{n=0}^{N-1}\int_0^1 D_n(t)dt = 1.$$

In contrast, $\int_0^1 |D_N(t)|dt \geq C\sum_{k=1}^{N}\frac{1}{k}$. Then, we can compute

$$|f(x)-\sigma_N(f)(x)| = \left|\int_0^1 \big(f(x)-f(x-t)\big)F_N(t)dt\right|$$

$$\leq \int_0^1 |f(x)-f(x-t)|F_N(t)dt.$$

(2) Let $\delta > 0$, then

$$\int_{\delta}^{1-\delta} F_N(t)dt = \frac{1}{N}\int_{\delta}^{1-\delta}\left(\frac{\sin(N\pi t)}{\sin(\pi t)}\right)^2 dt$$
$$\leq \frac{1}{N\sin^2(\delta\pi)} \to 0 \quad \text{as } N \to \infty.$$

Theorem 1.41. *Let* $\{F_N\}_{N\in\mathbb{N}} \in L^1([0,1])$ *be a sequence of functions such that* $F_N \geq 0$, $\forall N \in \mathbb{N}$ *and satisfy*

(i) $\int_0^1 F_N(t)dt = 1$,

(ii) *For all* $\delta > 0$, $\lim_{N\to\infty}\int_{\delta}^{1-\delta} F_N(t)dt = 0$.

Then,

(A) *For all* $p \in (1,\infty)$, $f \in L^p([0,1])$, *we have*

$$\lim_{N\to\infty}(f * F_N)(x) = f(x) \quad \text{in } L^p([0,1]).$$

(B) *If* $f \in C_{per}([0,1])$, *then*

$$\lim_{N\to\infty} f * F_N = f \quad \text{uniformly on } [0,1].$$

Proof. (A): Extend f periodically on $\mathbb{R}$, notice that

$$f(x) = \int_0^1 f(x)F_N(t)dt.$$

Then,

$$|(f * F_N)(x) - f(x)| = \left|\int_0^1 \big(f(x-t) - f(x)\big)F_N(t)dt\right|$$
$$\leq \int_0^1 |f(x-t) - f(x)|F_N(t)dt.$$

By Hölder's inequality, for $\frac{1}{p} + \frac{1}{p'} = 1$,

$$\int_0^1 |f(x-t) - f(x)|F_N(t)dt = \int_0^1 |f(x-t) - f(x)|\big(F_N(t)\big)^{\frac{1}{p}}\big(F_N(t)\big)^{\frac{1}{p'}}dt$$
$$\leq \left(\int_0^1 |f(x-t) - f(x)|^p F_N(t)dt\right)^{\frac{1}{p}}\left(\int_0^1 F_N(t)dt\right)^{\frac{1}{p'}}$$
$$= \left(\int_0^1 |f(x-t) - f(x)|^p F_N(t)dt\right)^{\frac{1}{p}}.$$

Thus,

$$\int_0^1 |(f * F_N)(x) - f(x)|^p dx \leq \int_0^1 \left(\int_0^1 |f(x-t) - f(x)|^p F_N(t)dt\right)dx.$$

By Fubini's theorem,

$$= \int_0^1 \left(\int_0^1 F_N(t)|f(x-t) - f(x)|^p dx\right)dt.$$

Let $\varepsilon > 0$. Pick $\delta > 0$ such that $|t| \leq \delta$ yields

$$\int_0^1 |f(x-t) - f(x)|^p dx < \frac{\varepsilon}{2}.$$

We can decompose $\int_0^1 \left(\int_0^1 |f(x-t)-f(x)|^p F_N(t)dt\right) dx$ as

$$\left(\int_\delta^{1-\delta} + \left(\int_0^\delta + \int_{1-\delta}^1\right)\right)\left(\int_0^1 |f(x-t)-f(x)|^p dx\right) F_N(t)dt.$$

Call these terms I and II.

For II:

$$\begin{aligned}\mathrm{II} &= \left(\int_0^\delta + \int_{1-\delta}^1\right)\left(\int_0^1 |f(x-t)-f(x)|^p dx\right) F_N(t)dt \\ &= \left(\int_0^\delta + \int_{-\delta}^0\right)\left(\int_0^1 |f(x-t)-f(x)|^p dx\right) F_N(t)dt \\ &< \varepsilon \int_0^1 F_N(t)dt \\ &= 2\varepsilon.\end{aligned}$$

Notice that for $\frac{1}{p}+\frac{1}{p'}=1$,

$$\begin{aligned}|a-b|^p &\le (|a|+|b|)^p \\ &\le \left((|a|^p+|b|^p)^{\frac{1}{p}} (1^{p'}+1^{p'})^{\frac{1}{p'}} \right)^p \\ &= 2^{\frac{p}{p'}}(|a|^p+|b|^p).\end{aligned}$$

Hence we have, by setting $a=f(x-t)$and $b=f(x)$, for I:

$$\begin{aligned}I &\le 2^{\frac{p}{p'}} \int_\delta^{1-\delta}\int_0^1 (|f(x-t)|^p+|f(x)|^p)\, dx\, F_N(t)dt. \\ &= 2^{\frac{p}{p'}+1}\|f\|^p_{L^p([0,1])}\int_\delta^{1-\delta} F_N(t)dt \to 0\end{aligned}$$

which $\to 0$ as $N\to\infty$. Hence,

$$\limsup_{N\to\infty}\int_0^1 |(f*F_N)(x)-f(x)|^p dx \le \varepsilon.$$

Since ε is arbitrary, we conclude that

$$\lim_{N\to\infty}\int_0^1 |(f*F_N)(x)-f(x)|^p dx = 0.$$

(B): Homework! Hint: Use

$$|f*F_N(x)-f(x)| \le \int_0^1 |f(x-t)-f(x)|F_N(t)dt.$$

□

Corollary 1.42 (Fejér's Theorem)**.** *We have the following:*

(A) *If $f\in L^p([0,1])$, $1<p<\infty$, then*

$$\lim_{N\to\infty}\sigma_N f = f \quad \textit{in } L^p([0,1]).$$

(B) *If $f\in C_{per}([0,1])$, then*

$$\lim_{N\to\infty}\sigma_N f = f \quad \textit{uniformly on } [0,1].$$

Proof. Notice

$$\begin{aligned}\sigma_N f(x) &= \int_0^1 f(x-t)\left[\frac{1}{N}\left(\frac{\sin N\pi t}{\sin \pi t}\right)^2\right] dt \\ &= (f*F_N)(x),\end{aligned}$$

and F_N satisfies (i) and (ii) of the previous theorem. □

Fejér's Theorem has the following deep consequence:

Theorem 1.43. $\{e_k\}_{k\in\mathbb{Z}} = \left\{e^{2\pi i k x}\right\}_{k\in\mathbb{Z}}$ *is an orthonormal basis of* $L^2([0,1])$.

Proof. Since

$$\begin{aligned}\langle e_m, e_n\rangle_{L^2([0,1])} &= \int_0^1 e^{2\pi i m x}\overline{e^{2\pi i n x}}\,dx \\ &= \int_0^1 e^{2\pi i (m-n)x}\,dx \\ &= \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}\end{aligned}$$

So $\{e_k\}_{k\in\mathbb{Z}}$ is orthonormal in $L^2([0,1])$. Recall from functional analysis that $\{e_k\}_{k\in\mathbb{Z}}$ is an orthonormal basis if and only if $S = \text{span}\{e_k\}_{k\in\mathbb{Z}}$ is dense. For $f \in C([0,1])$,

$$\begin{aligned}\sigma_N f(x) &= \frac{1}{N}\sum_{n=0}^{N-1} S_n f(x) \\ &= \frac{1}{N}\sum_{n=0}^{N-1}\left(\sum_{k=-n}^{n}\hat{f}(k)e^{2\pi i k x}\right) \\ &= S,\end{aligned}$$

and $\sigma_N f \to f$ in $L^2([0,1])$. So S is dense in $L^2([0,1])$. □

Corollary 1.44. *We then have the following*

(1) *If* $f \in L^2([0,1])$, *then*

$$f(x) = \sum_{k=-\infty}^{\infty}\hat{f}(k)e^{2\pi i k x}$$

in $L^2([0,1])$.

(2)

$$\|f\|^2_{L^2([0,1])} = \sum_{k=-\infty}^{\infty}|\hat{f}(k)|^2,$$

and the map

$$f \mapsto \{\hat{f}(k)\}_{k\in\mathbb{Z}}$$

is an isometric isomorphism from $L^2([0,1])$ *to* $\ell^2(\mathbb{Z})$.

Proof. (1): Notice that

$$\begin{aligned}f(x) &= \sum_{k=-\infty}^{\infty}\langle f, e_k\rangle e_k(x) \\ &= \sum_{k=-\infty}^{\infty}\hat{f}(k)e^{2\pi i k x}\end{aligned}$$

in $L^2([0,1])$.

(2): Use the fact that $\langle f, e_k\rangle_{L^2([0,1])} = \hat{f}(k)$ □

Remark 1.45. Let $\{F_N\}_{N\in\mathbb{N}} \in L^1(\mathbb{R}^d)$ be a sequence of functions such that $F_N \geq 0$, $\forall N \in \mathbb{N}$ and satisfy

(i) $\int_{\mathbb{R}^d} F_N(x)dx = 1$,

(ii) For all $\delta > 0$, $\lim_{N\to\infty} \int_{|x|>\delta} F_N(x)dx = 0$.

then

(A) $f * F_N \to f$ in $L^p(\mathbb{R}^d)$, $1 < p < \infty$.

(B) If $f \in C(\mathbb{R}^d)$ is bounded, $S \subseteq \mathbb{R}^d$ compact, then $f * F_N \to f$ uniformly in S.

1.4. **L^p convergence of Fourier Series.**

Proposition 1.46. *$S_N : L^p_{per}([0,1]) \to L^p_{per}([0,1])$ satisfies*

$$\|S_N\|_{op} \le 2N + 1.$$

Proof. Notice that

$$|S_N f(x)| \le \int_0^1 |f(x-t)||D_N(t)|\, dt$$

where $|D_N(t)| = \left|\sum_{k=-N}^{N} e^{2\pi ikx}\right|$, so

$$\begin{aligned} |S_N f(x)| &\le (2N+1)\int_0^1 |f(x-t)|\, dt \\ &\le (2N+1)\left(\int_0^1 |f(x-t)|^p\, dt\right)^{\frac{1}{p}} \\ &\le (2N+1)\|f\|_{L^p_{\text{per}}([0,1])}. \end{aligned}$$

□

Theorem 1.47 (Uniform Boundedness Principle)**.** *Let X be a Banach space, Y be a normed vector space, and $\{T_N\} \subseteq B(X,Y)$. If*

$$\sup_N \|T_N(x)\|_Y < \infty \quad \textit{for each } x \in X,$$

then

$$\sup_N \|T_N\|_{op} < \infty.$$

Lemma 1.48. *Let $1 \le p < \infty$, the following are equivalent:*

(1) $\lim_{N\to\infty} \|S_N f - f\|_{L^p_{per}([0,1])} = 0$ *for any* $f \in L^p_{per}([0,1])$.

(2) $\sup_N \|S_N\|_{op} = C_p < \infty$.

Proof. (1) $\implies$ (2) : We have $X = Y = L^p_{\text{per}}([0,1])$, $S_N \in B(L^p_{\text{per}}([0,1]))$. For $f \in L^p_{\text{per}}([0,1])$, pick $M = M(f)$ such that $N \ge M$ implies

$$\|S_N f - f\|_{L^p_{\text{per}}([0,1])} \le 1.$$

Then, for $1 \le N < M$,

$$\begin{aligned} \|S_N f\|_{L^p_{\text{per}}([0,1])} &\le (2N+1)\|f\|_{L^p_{\text{per}}([0,1])} \\ &\le (2M+1)\|f\|_{L^p_{\text{per}}([0,1])}. \end{aligned}$$

For $N \ge M$,

$$\begin{aligned} \|S_N f\|_{L^p_{\text{per}}([0,1])} &\le \|S_N f - f\|_{L^p_{\text{per}}([0,1])} + \|f\|_{L^p_{\text{per}}([0,1])} \\ &\le 1 + \|f\|_{L^p_{\text{per}}([0,1])}. \end{aligned}$$

Thus,

$$\sup_N \|S_n f\|_{L^p_{\text{per}}([0,1])} \le \max\left\{(2N+1)\|f\|_{L^p_{\text{per}}([0,1])}, 1 + \|f\|_{L^p_{\text{per}}([0,1])}\right\} < \infty.$$

Thus, by the Uniform Boundedness Principle,

$$\sup_N \|S_N\|_{\mathrm{op}} < \infty.$$

(2) $\implies$ (1) : By Fejér's theorem, $S = \operatorname{span}\left\{e^{2\pi i k x}\right\}_{k\in\mathbb{Z}}$is dense in $L^p_{\mathrm{per}}([0,1])$. Let $\varepsilon > 0$, pick $g \in S$ such that

$$\|f-g\|_{L^p_{\mathrm{per}}([0,1])} < \varepsilon.$$

If

$$g(x) = \sum_{l=-n}^{n} c_l e^{2\pi i l x},$$

we have

$$S_N g(x) = \sum_{k=-N}^{N} \widehat{g}(k) e^{2\pi i k x},$$

where

$$\begin{aligned}\widehat{g}(k) &= \sum_{l=-n}^{n} c_l \int_0^1 e^{2\pi i (l-k)x}\,dx \\ &= \begin{cases} c_k & \text{if } -n \le k \le n, \\ 0 & \text{if } |k| > n. \end{cases}\end{aligned}$$

Thus, for $N > n$,

$$S_N g(x) = \sum_{k=-n}^{n} c_k e^{2\pi i k x} = g(x).$$

Then

$$\begin{aligned}\|S_N f - f\|_{L^p_{\mathrm{per}}([0,1])} &\le \|S_N(f-g)\|_{L^p_{\mathrm{per}}([0,1])} + \|S_N g - f\|_{L^p_{\mathrm{per}}([0,1])} \\ &\le \|S_n\|_{op}\|g-f\|_{L^p_{\mathrm{per}}([0,1])} + \|g-f\|_{L^p_{\mathrm{per}}([0,1])} \\ &< (C_p+1)\varepsilon.\end{aligned}$$

Therefore, $\lim_{N\to\infty} \|S_N f - f\|_{L^p_{\mathrm{per}}([0,1])} = 0$. □

1.5. **Fourier Transform.** Here we start by giving a non-rigorous intuition of the Fourier transform as a limit of Fourier series. Clearly,

$$\int_{\mathbb{R}} |f(x)|^2\,dx = \lim_{l\to\infty} \int_{-l}^{l} |f(x)|^2\,dx.$$

Check that (homework!) if

$$e_k(x) = \frac{e^{\frac{\pi i k x}{L}}}{\sqrt{2l}},$$

then $\{e_k\}$ is an orthonormal basis for $L^2([-l,l])$. Thus,

$$\int_{-l}^{l} |f(x)|^2\,dx = \sum_{k=-\infty}^{\infty} \left| \underbrace{\frac{1}{\sqrt{2l}} \int_{-l}^{l} f(x) e^{-\frac{\pi i k x}{l}}\,dx}_{c_k = \langle f, e_k\rangle_{L^2([-L,L])}} \right|^2.$$

Informally, assume l is infinity gives

$$\frac{1}{2l} \sum_{k=-\infty}^{\infty} \left| \int_{\mathbb{R}} f(x) e^{-2\pi i \frac{k}{2l} x} dx \right|^2 dx.$$

Let

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} dx,$$

and

$$\xi_k = \frac{k}{2l}, \qquad \Delta\xi = \xi_k - \xi_{k-1} = \frac{1}{2l}.$$

Then,

$$\frac{1}{2l}\sum_{k=-\infty}^{\infty}\left|\int_{\mathbb{R}} f(x)e^{-2\pi i\frac{k}{2l}x}dx\right|^2 dx = \sum_{k=-\infty}^{\infty}\left|\widehat{f}(\xi_k)\right|^2\Delta x$$

which is a Riemann sum that converges to $\int_{-\infty}^{\infty}|\widehat{f}(\xi)|^2dx$ as $l \to \infty$. Hence, intuitively, we expect the Fourier transform to be an isometry on L^2.

For the Fourier inversion: For $f \in L^2(\mathbb{R})$, let $x \in \mathbb{R}$, $l > |x|$, define

$$f_l(x) = f(x)\chi_{[-l,l]}(x) = f(x) \in L^2([-l,l]).$$

Then

$$\begin{aligned} f_l(x) &= \sum_{k=-\infty}^{\infty}\langle f, e_k\rangle_{L^2([-l,l])}\frac{e^{\frac{\pi ikx}{l}}}{\sqrt{2l}} \\ &= \sum_{k=-\infty}^{\infty}\left(\frac{1}{\sqrt{2l}}\int_{-l}^{l} f(t)e^{-\frac{\pi ikt}{l}}dt\right)\frac{e^{\frac{2\pi ikx}{2l}}}{\sqrt{2l}}. \end{aligned}$$

Assume $l = \infty$ as before then we have

$$\begin{aligned} &\frac{1}{2l}\sum_{k=-\infty}^{\infty}\widehat{f}(\xi_k)e^{(2\pi i\xi_k)x} \\ &= \sum_{k=-\infty}^{\infty}\widehat{f}(\xi_k)e^{(2\pi i\xi_k)x}\Delta\xi \end{aligned}$$

which converges to $\int_{-\infty}^{\infty}\widehat{f}(\xi)e^{2\pi i\xi x}dx$ as $l \to \infty$.

Notice that by writing $\xi = -x$,

$$\mathcal{F}^{-1}(x) = \check{f}(x) = \widehat{f}(-x) = \int_{-\infty}^{\infty} f(u)e^{2\pi iux}du,$$

$$\mathcal{F}^{-1}(\widehat{f})(x) = f(x).$$

Lemma 1.49. *If $f(x) = e^{-\pi|x|^2}$, then $\widehat{f}(\xi) = e^{-\pi|\xi|^2}$.*

Proof. In $\mathbb{R}^n$,

$$\begin{aligned} \widehat{f}(\xi) &= \int_{\mathbb{R}^n} e^{-\pi|x|^2}e^{-2\pi i\xi\cdot x}\,dx \\ &= \int_{\mathbb{R}^n}\prod_{j=1}^{n} e^{-\pi x_j^2}e^{-2\pi i\xi_j x_j}\,dx_1\cdots dx_n \\ &= \left(\int_{\mathbb{R}} e^{-\pi x_1^2}e^{-2\pi i\xi_1 x_1}\,dx_1\right)\cdots\left(\int_{\mathbb{R}} e^{-\pi x_n^2}e^{-2\pi i\xi_n x_n}\,dx_n\right). \end{aligned}$$

Assume the lemma holds for $n = 1$, then,

$$\begin{aligned} \widehat{f}(\xi) &= e^{-\pi\xi_1^2}\cdots e^{-\pi\xi_n^2} \\ &= e^{-\pi|\xi|^2}. \end{aligned}$$

We now prove the case for $n = 1$, let $f(z) = e^{-\pi z^2}$, by complex analysis, using the following contour Γ,

$$0 = \int_{\Gamma} f(z)\,dz.$$

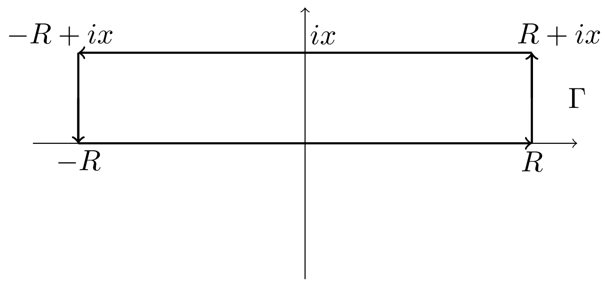

FIGURE 4. Contour Γ

Taking $R \to \infty$, we have

$$\lim_{R\to\infty} \int_{-R}^{R} e^{-\pi(t+ix)^2}\,dt = 1$$
$$= e^{\pi x^2} \lim_{R\to\infty} \int_{-R}^{R} e^{-\pi t^2} e^{-2\pi i x t}\,dt$$
$$= e^{\pi x^2} \widehat{f}(x).$$

Therefore, we proved the case where $n = 1$, i.e. $\widehat{f}(\xi) = e^{-\pi\xi^2}$. ☐

Remark 1.50. If $f_\lambda(x) = f\left(\frac{x}{\lambda}\right)$, then

$$\widehat{f_\lambda}(\xi) = \int_{\mathbb{R}^n} f\left(\frac{x}{\lambda}\right) e^{-2\pi i \xi\cdot x}\,dx.$$

By substitution,

$$\widehat{f_\lambda}(\xi) = \lambda^n \int_{\mathbb{R}^n} f(x) e^{-2\pi i(\lambda\xi)\cdot x}\,dx$$
$$= \lambda^n \widehat{f}(\lambda\xi).$$

Thus, if $g_\lambda(x) = e^{-\frac{\pi}{\lambda}|x|^2}$, then

$$\widehat{g_\lambda}(\xi) = \lambda^{\frac{n}{2}} \widehat{g_1}(\sqrt{\lambda}\xi)$$
$$= \lambda^{\frac{n}{2}} e^{-\pi\lambda|\xi|^2}.$$

Remark 1.51. Assume $F(x) \geq 0$ such that $\int_{\mathbb{R}^n} F(x)\,dx = 1$. For $\lambda > 0$, let

$$F_\lambda(x) = \lambda^n F(\lambda x).$$

Then, by substitution,

$$\int_{\mathbb{R}^n} F_\lambda(x)\,dx = 1.$$

One can check that (similar to the proof of Fejér's theorem)

$$f * F_\lambda \xrightarrow[\lambda\to\infty]{} f \quad \text{in } L^p(\mathbb{R}^n),$$

where

$$f * F_\lambda(x) = \int_{\mathbb{R}^n} f(x-y)\,F_\lambda(y)\,dy.$$

Theorem 1.52 (Parseval's theorem and Fourier Inversion)**.** *Our immediate goals are, for* $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$,

(1) $\|f\|_{L^2(\mathbb{R})} = \|\widehat{f}\|_{L^2(\mathbb{R})}$

(2) $f(x) = \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{2\pi i \xi x} d\xi,\ \mathcal{F}^{-1}(\hat{f}(x))$ *a.e.*

Proof. (1): By Monotone Convergence Theorem,

$$\int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 \, d\xi = \lim_{j\to\infty} \int_{\mathbb{R}^n} |f(u)|^2 e^{-\frac{\pi}{j}|u|^2} \, du$$

Since

$$\left|\widehat{f}(u)\right|^2 = \widehat{f}(u)\overline{\widehat{f}(u)},$$

we have

$$\begin{aligned}\int_{\mathbb{R}^n} |f(u)|^2 e^{-\frac{\pi}{j}|u|^2} \, du &= \int_{\mathbb{R}^{3n}} f(x)\overline{f(y)} e^{-\frac{\pi}{j}|u|^2} e^{-2\pi i(x-y)\cdot u} \, dx \, dy \, du \\ &= \int_{\mathbb{R}^{2n}} f(x)\overline{f(y)} \left(\int_{\mathbb{R}} e^{-\frac{\pi}{j}|u|^2} e^{-2\pi i(x-y)\cdot u} \, du \right) dx \, dy \\ &= j^{\frac{n}{2}} \int_{\mathbb{R}^{2n}} f(x)\overline{f(y)} e^{-\pi j|x-y|^2} \, dx \, dy \\ &= \int_{\mathbb{R}^n} f(x) \overline{\left(j^{\frac{n}{2}} \int_{\mathbb{R}^n} f(y) e^{-\pi j|x-y|^2} \, dy \right)} dx \\ &= \int_{\mathbb{R}^n} f(x)\overline{(f * F_{\sqrt{j}})(x)} \, dx.\end{aligned}$$

As $j \to \infty$, we obtain

$$\lim_{j\to\infty} \int_{\mathbb{R}^n} |f(u)|^2 e^{-\frac{\pi}{j}|u|^2} \, du = \int_{\mathbb{R}^n} |f(x)|^2 \, dx.$$

(2): Let

$$g_j(x) = e^{-\frac{\pi}{j}|x|^2},$$

then

$$\widehat{g}_j(x) = j^{n/2} e^{-\pi j|x|^2}.$$

Therefore,

$$\begin{aligned}\widehat{g}_j(x-y) &= j^{n/2} e^{-\pi j|x-y|^2} \\ &= F_{\sqrt{j}}(x-y).\end{aligned}$$

Since $f * F_{\sqrt{j}} \to f$ in $L^2(\mathbb{R}^n)$, as $j \to \infty$, pick a subsequence $\{j_k\}$ that converges a.e. So for a.e. $x \in \mathbb{R}^n$,

$$f(x) = \lim_{k\to\infty} \int_{\mathbb{R}^n} f(y) F_{\sqrt{j_k}}(x-y) \, dy.$$

Notice that

$$\begin{aligned}\int_{\mathbb{R}^n} f(y) F_{\sqrt{j_k}}(x-y) \, dy &= \int_{\mathbb{R}^n} f(y) \left(\int_{\mathbb{R}^n} g_{j_k}(u) e^{2\pi i(x-y)\cdot u} \, du \right) dy \\ &= \int_{\mathbb{R}^n} g_{j_k}(u) \left(\int_{\mathbb{R}^n} f(y) e^{2\pi i(-u)y} \, dy \right) e^{2\pi i x\cdot u} du \\ &= \int_{\mathbb{R}^n} g_{j_k}(u) \hat{f}(u) e^{2\pi i x\cdot u} du \\ &= \mathcal{F}(g_{j_k}\hat{f})(-x).\end{aligned}$$

Finally, by dominated convergence theorem,

$$g_{j_k}\hat{f} \to \hat{f} \text{ in } L^2(\mathbb{R}^n),$$

because

$$\begin{aligned}|g_{j_k}\hat{f} - \hat{f}|^2 &\le (|g_{j_k}\hat{f}| + |\hat{f}|)^2 \\ &\le (2|\hat{f}|)^2 = 4|\hat{f}|^2.\end{aligned}$$

Therefore,

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} \left| \mathcal{F}(g_{j_k}\hat{f})(-x) - \mathcal{F}(\hat{f})(-x) \right|^2 dx = \lim_{k\to\infty} \int_{\mathbb{R}^n} \left| \mathcal{F}\left[(g_{j_k}\hat{f})(x) - \hat{f}(x) \right] \right|^2 dx.$$

Since $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is isometric,

$$= \lim_{k\to\infty} \int_{\mathbb{R}^n} \left| g_{j_k} \hat{f}(x) - \hat{f}(x) \right|^2 dx = 0.$$

□

1.6. **Fourier Transform on L^2.**

Lemma 1.53. *Let $T : X \to Y$, where Y is a Banach space. Suppose $T|_U \to Y$ is isometric, where $U \subseteq X$ is a dense subspace. Then T uniquely extends to an isometry $T : X \to Y$.*

Proof. Let $x \in X$ and $\{x_k\} \subseteq U$ with $x_k \to x$. Then,

$$\|Tx_k - Tx_m\|_Y = \|T(x_k - x_m)\|_Y$$
$$= \|x_k - x_m\|_X$$

Hence, $\{Tx_k\}$ is Cauchy. Let

$$\widetilde{T}x = \lim_{k\to\infty} Tx_k.$$

To complete the proof, one checks the following claims

(1) $\widetilde{T}x$ is independent of $\{x_k\}$ converging to x.

(2) $\widetilde{T} : X \to Y$ is a linear isometry.

(3) $\widetilde{T}x = Tx$ if $x \in U$.

(4) $\widetilde{T}$ is the unique linear isometry satisfying (3).

□

Definition 1.54. For Fourier transform in L^2, we have $X = Y = L^2(\mathbb{R}^n)$, $T = \mathcal{F}$, $U = L^1 \cap L^2$. If $f \in L^2(\mathbb{R}^n)$ define its Fourier transform as

$$\widehat{f} = \lim_{k\to\infty} \widehat{f_k}$$

for any $\{f_k\} \subseteq L^1 \cap L^2$ where $f_k \to f$ in $L^2(\mathbb{R}^n)$.

Also define the inverse Fourier transform

$$\check{f}(x) = \widehat{f}(-x).$$

Proposition 1.55. *Note that*

(1) $\|\widehat{f}\|_{L^2(\mathbb{R}^n)} = \|\check{f}\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}$.

(2) $f = \mathcal{F}^{-1}(\widehat{f})$ *a.e.*

Proof. (2): Notice that as $k \to \infty$,

$$f_k = (\widehat{f_k})^\vee \to (\widehat{f})^\vee$$

and

$$f_k \to f,$$

so $f = (\widehat{f})^\vee$ a.e. □

Remark 1.56. Conclusion: $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is unitary.

Example 1.57. Let $f \in C_0^\infty(\mathbb{R})$, then

$$\mathcal{F}(f')(\xi) = \lim_{R\to\infty} \int_{-R}^{R} f'(x) e^{-2\pi i \xi x} dx$$
$$= \lim_{R\to\infty} \left[e^{-2\pi i \xi x} f(x) \Big|_{-R}^{R} - (2\pi i \xi) \int_{-R}^{R} f'(x) e^{-2\pi i \xi x} dx \right]$$
$$= 2\pi i \xi \widehat{f}(\xi).$$

Similarly, if $f \in C_0^\infty(\mathbb{R}^n)$, $\alpha \in \mathbb{N}_{\geq 0}^n$,

$$\mathcal{F}\left(\partial^\alpha f\right)(\xi) = (2\pi i)^{|\alpha|} \xi^\alpha \widehat{f}(\xi).$$

Example 1.58. The Fourier transform of the laplacian is

$$\mathcal{F}(\Delta f)(\xi) = \sum_{j=1}^n \widehat{\frac{\partial^2 f}{\partial x_j^2}} = -4\pi^2 \sum_{j=1}^n \xi_j^2 \widehat{f}(x) = -4\pi^2 |\xi|^2 \widehat{f}(\xi).$$

Proposition 1.59. $\widehat{u * v} = \widehat{u}\,\widehat{v}$.

Proof. Homework! □

Example 1.60. To solve $-\Delta u + u = f$, we start by taking the Fourier transform

$$4\pi^2 |x|^2 \widehat{u}(\xi) + \widehat{u}(\xi) = \widehat{f}(\xi).$$

Then, one can solve that

$$\widehat{u}(\xi) = \frac{1}{4\pi^2} \left(\frac{\widehat{f}(\xi)}{1 + \frac{1}{4\pi^2}|\xi|^2} \right)$$
$$u = \mathcal{F}^{-1} \left(\frac{1}{4\pi^2} \left(\frac{\widehat{f}(\xi)}{1 + \frac{1}{4\pi^2}|\xi|^2} \right) \right).$$

Set the Bessel potential $B = \mathcal{F}^{-1}\left(\frac{1}{1+\frac{1}{4\pi^2}|\xi|^2} \right)$. Then $\widehat{B} = \frac{1}{1+\frac{1}{4\pi^2}|\xi|^2}$, so

$$u(x) = \frac{1}{4\pi^2} \mathcal{F}^{-1}\left(\widehat{f}\,\widehat{B} \right)$$
$$= \frac{1}{4\pi^2} \mathcal{F}^{-1}(\widehat{f * B})$$
$$= \frac{1}{4\pi^2}(f * B).$$

One can check that this equals

$$u(x) = \frac{1}{(4\pi)^{n/2}} \int_0^\infty \int_{\mathbb{R}^n} \frac{e^{-t-\frac{|x-y|^2}{4t}}}{t^{n/2}} f(y)\, dy\, dt.$$

1.7. **Fourier Transform on L^p, $1 < p \leq 2$.** Notice that for $f \in L^p([0,1])$, $1 \leq p \leq \infty$

$$|\hat{f}(\xi)| = \left| \int_0^1 f(x) e^{-2\pi i \xi x} dx \right|$$
$$\leq \int_0^1 |f(x)| dx$$
$$\leq \|f\|_{L^p([0,1])}.$$

In order to define Fourier transform $\hat{f}(x)$ for $f \in L^p(\mathbb{R}^n)$, $1 < p, q < \infty$, we need the following

(1) Prove $\|\widehat{f}\|_{L^q(\mathbb{R}^n)} \leq C_{p,q} \|f\|_{L^p(\mathbb{R}^n)}$, for $f \in L^1 \cap L^p$.

(2) Define $\widehat{f} = \lim_{n\to\infty} \widehat{f_n}$, where $f_n \in L^1 \cap L^p$, $f_n \to f$ in L^p, so $\widehat{f} \in L^q$.

Remark 1.61. Important: If (1) is true, then $q = p' = \frac{p}{p-1}$. Why? Scaling argument!

Proof. Recall: If $f_\lambda(x) = f\left(\frac{x}{\lambda}\right)$ then

$$\begin{aligned}\widehat{f_\lambda}(\xi) &= \int_{\mathbb{R}^n} f\left(\frac{x}{\lambda}\right) e^{-2\pi i(\xi\cdot x)} dx \\ &= \lambda^n \int_{\mathbb{R}^n} f(x) e^{-2\pi i(\lambda\xi)\cdot x} dx \\ &= \lambda^n \widehat{f}(\lambda\xi),\end{aligned}$$

so

$$\widehat{f}(\lambda\xi) = \lambda^{-n}\widehat{f_\lambda}(\xi).$$

Therefore, assuming (1) is true, $f \in L^1 \cap L^p$,

$$\begin{aligned}\left[\int_{\mathbb{R}^n} |\widehat{f_\lambda}(\xi)|^q d\xi\right]^{1/q} &= \left[\int_{\mathbb{R}^n} |\lambda^n \widehat{f}(\lambda\xi)|^q d\xi\right]^{1/q} \\ &= \lambda^{n/q}\left[\int_{\mathbb{R}^n} |\lambda^{-n}\widehat{f_\lambda}(\xi)|^q d\xi\right]^{1/q} \\ &= \lambda^{n\left(\frac{1}{q}-1\right)} \|\widehat{f_\lambda}\|_{L^q} \\ &\le C_{p,q}\lambda^{n\left(\frac{1}{q}-1\right)} \|f_\lambda\|_{L^p} \\ &= C_{p,q}\lambda^{n\left(\frac{1}{q}-1\right)} \left[\int_{\mathbb{R}^n} \left|f\left(\frac{x}{\lambda}\right)\right|^p dx\right]^{\frac{1}{p}} \\ &= C_{p,q}\lambda^{n\left(\frac{1}{q}-1\right)} \lambda^{\frac{n}{p}} \left[\int_{\mathbb{R}^n} |f(x)|^p dx\right]^{\frac{1}{p}}.\end{aligned}$$

Let $-1 + \frac{1}{p} = \frac{1}{p'}$, we conclude that for any $f \in L^1 \cap L^p$,

$$\|\widehat{f}\|_{L^q} \le C_{p,q}\lambda^{n\left(\frac{1}{q}-\frac{1}{p'}\right)} \|f\|_{L^p}.$$

Assume $q < p'$, let $\lambda \to 0^+$, then $\|\widehat{f}\|_{L^q} \le 0$, which implies $\widehat{f} \equiv 0$ a.e. for all $f \in L^1 \cap L^p$. Similarly, if $q > p'$, let $\lambda \to \infty$, we have $\widehat{f} \equiv 0$ a.e. So $q \ne p'$ implies $\hat{f} \equiv 0$ a.e. Absurd! (example: $f(x) = e^{-\pi|x|^2} = \widehat{f}(x)$).

Therefore, if (1) is true then $q = p'$. ◻

Remark 1.62. (1) is false if $p > 2$.

Proposition 1.63. *If p is between p_0 and p_1, with $p_0 \ne p_1$, and $f \in L^{p_0} \cap L^{p_1}$, then $f \in L^p$.*

Proof. Without loss of generality, assume $p_0 < p < p_1 < \infty$, let

$$\theta = \frac{\frac{1}{p_0} - \frac{1}{p}}{\frac{1}{p_0} - \frac{1}{p_1}} \in (0,1).$$

So

$$\begin{aligned}\frac{1}{p} &= \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \\ 1 &= \frac{(1-\theta)p}{p_0} + \frac{\theta p}{p_1}.\end{aligned}$$

Then

$$\int_{\mathbb{R}^n} |f(x)|^p dx = \int_{\mathbb{R}^n} |f(x)|^{(1-\theta)p} |f(x)|^{\theta p} dx.$$

$$\leq \left[\int_{\mathbb{R}^n} |f(x)|^{p_0} dx\right]^{\frac{(1-\theta)p}{p_0}} \left[\int_{\mathbb{R}^n} |f(x)|^{p_1} dx\right]^{\frac{\theta p}{p_1}}.$$

Thus,

$$\left[\int_{\mathbb{R}^n} |f(x)|^{p} dx\right]^{\frac{1}{p}} \leq \left[\int_{\mathbb{R}^n} |f(x)|^{p_0} dx\right]^{\frac{1-\theta}{p_0}} \left[\int_{\mathbb{R}^n} |f(x)|^{p_1} dx\right]^{\frac{\theta}{p_1}}.$$

So

$$\|f\|_{L^p} \leq \|f\|_{L^{p_0}}^{1-\theta} \|f\|_{L^{p_1}}^{\theta}.$$

The case where $p_1 = \infty$ is left as homework. □

Definition 1.64. Let

$$F = \{f : f \text{ simple, has finite measure support}\} \subseteq \bigcap_{p>0} L^p(\mathbb{R}^n).$$

Let M be the set of measurable functions on $\mathbb{R}^n$.

Theorem 1.65 (Riesz-Thorin Interpolation)**.** *Let $0 < p_0, p_1 \leq \infty$, $1 \leq q_0, q_1 \leq \infty$. Fix $0 < \theta < 1$, and let*

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Let $T : F \to M$ be a linear operator such that $(Tf) \cdot g \in L^1$ for $f, g \in F$, and satisfies

$$\|Tf\|_{L^{q_0}} \leq A_0 \|f\|_{L^{p_0}}, \quad \|Tf\|_{L^{q_1}} \leq A_1 \|f\|_{L^{p_1}}, \quad \forall f \in F.$$

Then for all such f,

$$\|Tf\|_{L^{q_\theta}} \leq A_0^{1-\theta} A_1^{\theta} \|f\|_{L^{p_\theta}}.$$

Proof. Surprisingly, the proof follows from the Maximum Modulus Principle. See A. □

Theorem 1.66 (Hausdorff-Young Inequality)**.** *If $1 < p \leq 2$ then $\mathcal{F}$ uniquely extends from $L^1 \cap L^p$ to L^p with*

$$\|\widehat{f}\|_{L^{p'}(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)}.$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. Key tool: interpolation! Let $q_0 = p_0 = 2$, $p_1 = 1$, $q_1 = \infty$. Let $T = \mathcal{F}$. Then, for $f \in F$, $f \in L^1 \cap L^2$, we have

$$\|\widehat{f}\|_{L^{q_0}} = \|\widehat{f}\|_{L^2} = \|f\|_{L^2} = \|f\|_{L^{p_0}},$$

and

$$\begin{aligned} \|\widehat{f}\|_{L^{q_1}} &= \|\widehat{f}\|_{L^\infty} \\ &= \left\|\int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x}\, dx\right\|_{L^\infty} \\ &\leq \int_{\mathbb{R}^n} |f(x)|\, dx \\ &= \|f\|_{L^1} \\ &= \|f\|_{L^{p_1}}. \end{aligned}$$

Thus, by Riesz-Thorin Interpolation, for $0 < \theta < 1$ fixed,

$$\|\widehat{f}\|_{L^{q_\theta}} \leq \|f\|_{L^{p_\theta}},$$

where

$$\frac{1}{p_\theta} = \frac{1-\theta}{2} + \theta = \frac{1}{2} + \frac{\theta}{2}, \quad \frac{1}{q_\theta} = \frac{1-\theta}{2}.$$

Setting $p_\theta = p$, we can solve for θ:

$$\theta = 2\left(\frac{1}{p} - \frac{1}{2}\right) = \frac{2}{p} - 1 \in (0,1)$$

$$\frac{1}{q_\theta} = \frac{1-\theta}{2} = 1 - \frac{1}{p} = \frac{1}{p'}$$

So $q_\theta = p'$, hence

$$\|\widehat{f}\|_{L^{p'}} \leq \|f\|_{L^p}, \quad \text{for } f \in F.$$

The rest follows from a density argument, and is left as homework. □

1.8. **Schwartz Space.** A vector $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ is called a multi-index of order $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We denote

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}},$$

and similarly for $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. $\alpha \leq \beta$ means $\alpha_i \leq \beta_i$, $i \in \{1, \dots, n\}$. Define for $\alpha \geq \beta$,

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}.$$

Note that $|\alpha| = k+1$ implies $D^\alpha = D^{\alpha'} \frac{\partial}{\partial x_j}$, where $|\alpha'| = k$, $\alpha = \alpha' + e_j$.

Theorem 1.67 (Leibniz rule)**.**

$$D^\alpha(fg) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta f D^{\alpha-\beta} g$$

Definition 1.68. The Schwartz space is defined as

$$\mathcal{S}(\mathbb{R}^n) = \left\{ f \in C^\infty(\mathbb{R}^n) : \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)| < \infty, \ \forall \alpha, \beta \in \mathbb{N}_0^n \right\}.$$

Also define the operator

$$\sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)| = p_{\alpha,\beta}(f).$$

Remark 1.69. Note that for $\forall f \in \mathcal{S}(\mathbb{R}^n)$, $\forall \beta \in \mathbb{N}_0^n$, $\forall k \in \mathbb{N}$, we have

$$\lim_{|x| \to \infty} |x|^k \left|(D^\alpha f)(x)\right| = 0.$$

Example 1.70. Here is a few exmaple about Schwartz space:

$$e^{-|x|^2} = e^{-(x_1^2 + \cdots + x_n^2)} \in \mathcal{S}(\mathbb{R}^n),$$

$$C_0^\infty(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^m),$$

$$e^{-x^2} \sin(e^{x^2}) \notin \mathcal{S}(\mathbb{R}^n).$$

Definition 1.71. For the metrix on $\mathcal{S}(\mathbb{R}^n)$, let $\{(\alpha(k), \beta(k))\}_{k=1}^\infty \subseteq \mathbb{N}_0^n \times \mathbb{N}_0^n$ be the enumeration of pairs of multi-indices. Define

$$d(f,g) = \sum_{k=0}^\infty 2^{-k} \frac{p_{\alpha(k),\beta(k)}(f-g)}{1 + p_{\alpha(k),\beta(k)}(f-g)}.$$

Proposition 1.72. *If $f_m \to f$ in $(\mathcal{S}(\mathbb{R}^n), d)$ if and only if*

$$p_{\alpha,\beta}(f - f_m) \to 0 \quad \textit{for any } \alpha, \beta \in \mathbb{N}_0^n.$$

Remark 1.73. If $f \in \mathcal{S}(\mathbb{R}^n)$, then $x^\alpha f(x) \in \mathcal{S}(\mathbb{R}^n)$ for any multi-index α.

Lemma 1.74. *Let $f \in \mathcal{S}(\mathbb{R}^n)$, then*

(a) $\widehat{(D^\alpha f)}(\xi) = (2\pi i \xi)^{|\alpha|} \widehat{f}(\xi)$

(b) $D^\alpha \widehat{f}(\xi) = (-2\pi i)^{|\alpha|}\mathcal{F}\left[(x)^\alpha f(x)\right](\xi)$

The same is true for $\mathcal{F}^{-1}$, but with no negative sign in (b).

Proof. (a); Base case $|\alpha| = 0$: nothing to prove. Assume true for $|\alpha| = k$, for any $f \in \mathcal{S}(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$. Let $|\alpha| = k+1$, write $D^\alpha = D^{\alpha'}\frac{\partial}{\partial x_j}$, where $|\alpha'| = k$, $\alpha = \alpha' + e_j$. Then,

$$\begin{aligned}\widehat{(D^\alpha f)}(\xi) &= \mathcal{F}\left(D^{\alpha'}\frac{\partial}{\partial x_j}f\right)(\xi)\\ &= (2\pi i)^{|\alpha'|}\xi^{|\alpha'|}\mathcal{F}\big(\frac{\partial}{\partial x_j}f\big)(\xi).\end{aligned}$$

Since $\mathcal{F}\left(\frac{\partial f}{\partial x_j}\right)(\xi) = 2\pi i\xi_j\widehat{f}(x)$, we have

$$\begin{aligned}\widehat{(D^\alpha f)}(\xi) &= (2\pi i)^{(|\alpha'|+1)}\xi^{\alpha'}\xi_j\widehat{f}(\xi)\\ &= (2\pi i)^{|\alpha|}\xi^\alpha\hat{f}(\xi).\end{aligned}$$

Now we prove $\mathcal{F}\left(\frac{\partial f}{\partial x_j}\right)(\xi) = 2\pi i\xi_j\widehat{f}(x)$, as follows, notice by definition

$$\begin{aligned}\mathcal{F}\left(\frac{\partial f}{\partial x_j}\right)(\xi) &= \int_{\mathbb{R}^n}\frac{\partial f}{\partial x_j}(x)\,e^{-2\pi i\xi\cdot x}\,dx\\ &= \int_{\mathbb{R}^n}\frac{\partial f}{\partial x_j}(x)\,e^{-2\pi i\left[\sum_{l=1}^n \xi_l x_l\right]}\,dx\\ &= \lim_{R\to\infty}\int_{\mathbb{R}^{n-1}}\left[\prod_{\substack{l=1,\\ l\neq j}}^n\int_{-R}^R\frac{\partial f}{\partial x_j}(x_1,\dots,x_j,\dots,x_n)e^{-2\pi i\xi_j\cdot x_j}\,dx_j\right]d\tilde{x}\end{aligned}$$

where $\tilde{x} = (x_1,\dots,x_{j-1},x_{j+1},\dots,x_n) \in \mathbb{R}^{n-1}$. Then, by integration by parts, the boundary term vanishes, so

$$\begin{aligned}\mathcal{F}\left(\frac{\partial f}{\partial x_j}\right)(\xi) &= 2\pi i\xi_j\int_{\mathbb{R}^{n-1}}\left[\prod_{\substack{l=1,\\ l\neq j}}^n e^{-2\pi i\xi_l\cdot x_l}\int_{-R}^R\frac{\partial f}{\partial x_j}(x)e^{-2\pi i\xi_j\cdot x_j}\,dx_j\right]d\tilde{x}\\ &= 2\pi i\xi_j\int_{\mathbb{R}^{n-1}}\left[\int_{-\infty}^\infty f(x)\prod_{l=1}^n e^{-2\pi i\xi_j x_j}\,dx_j\right]d\tilde{x}\\ &= 2\pi i\xi_j\int_{\mathbb{R}^n}f(x)e^{-2\pi i\xi\cdot x}\,dx\\ &= 2\pi i\xi_j\widehat{f}(x).\end{aligned}$$

□

Proposition 1.75. *$\mathcal{F}: \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a bijection.*

Proof. (b) implies $\widehat{f} \in C^\infty(\mathbb{R}^n)$, by (b) and (a),

$$\begin{aligned}\xi^\alpha(D^\beta\widehat{f})(\xi) &= (2\pi i)^{|\beta|}\,\xi^\alpha\mathcal{F}\left[(x)^\beta f(x)\right](\xi)\\ &= \mathcal{F}\left[D^\alpha\left((x)^\beta f(x)\right)\right](\xi).\end{aligned}$$

Thus,

$$\begin{aligned}\|(\xi)^\alpha(D^\beta\widehat{f})(\xi)\|_{L^\infty} &= \left\|D^\alpha\mathcal{F}\left[(x)^\beta f(x)\right]\right\|_{L^\infty}\\ &\le \|D^\alpha\big((x)^\beta f(x)\big)\|_{L^1} < \infty.\end{aligned}$$

Hence, $\widehat{f} \in \mathcal{S}(\mathbb{R}^n)$. Similarly, $f \in \mathcal{S}(\mathbb{R}^n)$ implies $\widehat{f} \in \mathcal{S}(\mathbb{R}^n)$ and $\mathcal{F}^{-1}$ is the inverse of the Fourier transform on $\mathcal{S}(\mathbb{R}^n) \subseteq L^1 \cap L^2$.

□

Proposition 1.76. *$f_m \to f$ in $\big(\mathcal{S}(\mathbb{R}^n), d\big)$ if and only if $\widehat{f_m} \to \widehat{f}$ in $\big(\mathcal{S}(\mathbb{R}^n), d\big)$.*

Proof. Homework! □

2. The Hilbert Transform

2.1. L^p Convergence of Fourier Transform and the Hilbert Transform.

Proposition 2.1. *Let* $f_R(x) = \chi_{B(0,R)}(x)\widehat{f}(x)$, $f \in L^2$. *Then*

$$\lim_{R\to\infty} \mathcal{F}^{-1}\left(\chi_{B(0,R)}\widehat{f_R}\right) = f \quad \text{in } L^2.$$

Proof.

$$\|\mathcal{F}^{-1}\left(\chi_{B(0,R)}\widehat{f_R}\right) - \mathcal{F}^{-1}(\widehat{f})\|_{L^2} = \|\chi_{B(0,R)}\widehat{f_R} - \widehat{f}\|_{L^2}$$

$$\leq \|\chi_{B(0,R)}\widehat{f_R} - \chi_{B(0,R)}\widehat{f}\|_{L^2} + \|\chi_{B(0,R)}\widehat{f} - \widehat{f}\|_{L^2}$$

which $\to 0$ as $R \to \infty$ by the dominated convergence theorem since

$$\|\chi_{B(0,R)}\widehat{f} - \widehat{f}\|_{L^2} \leq \|\widehat{(f_R - f)}\|_{L^2} = \|f_R - f\|_{L^2}$$

which $\to 0$ as $R \to \infty$. ☐

Remark 2.2. Notice that $f \in L^p$ $(p \geq 1)$ implies $f_R \in L^1$ implies $\widehat{f_R} \in L^\infty$ implies $\chi_{B(0,R)}\widehat{f_R} \in \bigcap_{q>0} L^q$.

Question 2.3. If $f \in L^p(\mathbb{R}^d)$, $p > 1$, then does

$$\lim_{R\to\infty} \mathcal{F}^{-1}(\chi_{B(0,R)}\widehat{f_R}) = f \quad \text{in } L^p(\mathbb{R}^d)?$$

Answer: Yes, if $d = 1$. No, if $d > 1$.

Theorem 2.4 (Young's inequality)**.** *If* $f \in L^p$, $g \in L^1$, *then*

$$\|f * g\|_{L^p} \leq \|f\|_{L^p}\|g\|_{L^1}.$$

Proof. Homework! ☐

Lemma 2.5. *If* $f \in L^2$, $g \in L^1$, *then*

$$\widehat{f * g} = \hat{f}\hat{g}.$$

Proof. Let $f_n \in L^1 \cap L^2$, $f_n \to f$ in L^2, then $\widehat{f} = \lim_{n\to\infty} \widehat{f_n}$ in L^2. Thus

$$\hat{f}\hat{g} = \lim_{n\to\infty} \hat{f}_n\widehat{g} \quad \text{in } L^2$$

$$\lim_{n\to\infty} \widehat{f_n * g}$$

$$= \widehat{f * g} \quad \text{in } L^2.$$

Also,

$$\|\widehat{f_n * g} - \widehat{f * g}\|_{L^2} = \|f_n * g - f * g\|_{L^2}$$

$$= \|(f_n - f) * g\|_{L^2}$$

$$\leq \|f_n - f\|_{L^2}\|g\|_{L^1}$$

which $\to 0$ as $n \to \infty$. ☐

Definition 2.6. Define the partial sum operator

$$S_R g = \mathcal{F}^{-1}\left(\chi_{B(0,R)}\hat{g}\right), \quad g \in L^1.$$

Then we are interested in whether

$$\lim_{R\to\infty} S_R f = f.$$

Notice

$$\begin{aligned}\mathcal{F}^{-1}\left(\chi_{B(0,R)}\hat{g}\right) &= \mathcal{F}^{-1}\left(\mathcal{F}\left[\mathcal{F}^{-1}(\chi_{B(0,R)})\right]\hat{g}\right)\\ &= \mathcal{F}^{-1}\left[\mathcal{F}\left(\mathcal{F}^{-1}(\chi_{B(0,R)}) * g\right)\right]\\ &= \mathcal{F}^{-1}(\chi_{B(0,R)}) * g.\end{aligned}$$

For $d = 1$,

$$\begin{aligned}\mathcal{F}^{-1}\left(\chi_{B(0,R)}\right)(x) &= \int_{-R}^{R} e^{2\pi i x\xi}\, d\xi\\ &= \frac{e^{2\pi i R x} - e^{-2\pi i R x}}{2\pi i x}\\ &= \frac{\sin(2\pi R x)}{\pi x}, \text{ which is similar to the Dirichlet kernel}\\ &= D_R(x).\end{aligned}$$

So

$$S_R g(x) = \int_{-\infty}^{\infty} g(y)\frac{\sin\big(2\pi R(x-y)\big)}{\pi(x-y)}\, dy.$$

Remark 2.7. Note that

$$\begin{aligned}\|S_R g\|_{L^p} &\le \|g\|_{L^1}\|D_R\|_{L^p}\\ &\le C_{R,p}\|g\|_{L^1}.\end{aligned}$$

Remark 2.8. Notice that

$$\begin{aligned}\|S_R g\|_{L^p}^p &= \int_{\mathbb{R}}\left|\int_{\mathbb{R}} g(x-y)\frac{\sin(2\pi R y)}{\pi y}\, dy\right|^p dx\\ &= \int_{\mathbb{R}}\left|\int_{\mathbb{R}} g\left(x-\frac{y}{R}\right)\frac{\sin(2\pi y)}{\pi y}\, dy\right|^p dx\\ &= \frac{1}{R}\int_{\mathbb{R}}\left|\int_{\mathbb{R}} g\left(\frac{x-y}{R}\right)\frac{\sin(2\pi y)}{\pi y}\, dy\right|^p dx.\end{aligned}$$

Let

$$\left[g\left(\frac{\cdot}{R}\right)\right](x) = g\left(\frac{x}{R}\right).$$

Then

$$\begin{aligned}\|S_R g\|_{L^p}^p &= \frac{1}{R}\left\|S_1 g\left(\frac{\cdot}{R}\right)\right\|_{L^p}^p\\ &\le \|S_1\|_{L^p\cap L^1\to L^p}\int_{\mathbb{R}}\left|g\left(\frac{x}{R}\right)\right|^p dx\\ &= \|S_1\|_{\text{op}}\|g\|_{L^p}.\end{aligned}$$

Lemma 2.9. $\lim_{R\to\infty} S_R f = f$ *for* $f \in L^p$, $1 \le p < \infty$, *if* $\|S_1\|_{op} = C_p < \infty$.

Proof. Let $\varepsilon > 0$. Pick $\phi \in \mathcal{S}$ such that

$$\|f-\phi\|_{L^p} < \frac{\varepsilon}{1+C_p}.$$

Then

$$\begin{aligned}\limsup_{R\to\infty}\|S_R f_R - f\|_{L^p} &\le \limsup_{R\to\infty}\left[\|\underbrace{S_R f_R - S_R\phi}_{S_R(f_R-\phi)}\|_{L^p} + \underbrace{\|S_R\varphi - \phi\|_{L^p}}_{\to 0 \text{ as } R\to\infty} + \|\phi - f\|_{L^p}\right].\\ &\le \limsup_{R\to\infty}\left[C_p\|f-\phi\|_{L^p} + \frac{\varepsilon}{1+C_p}\right].\end{aligned}$$

By the Dominated Convergence Theorem,

$$\limsup_{R\to\infty} \|S_R f_R - f\|_{L^p} \leq \frac{\varepsilon C_p}{1+C_p} + \frac{\varepsilon}{1+C_p} = \varepsilon.$$

□

Remark 2.10. S_1 does not map L^1 into L^1.

Proof. Let

$$f(x) = \chi_{(-\frac{1}{12},\frac{1}{12})}(x),$$

then

$$\begin{aligned} S_1 f(x) &= \int_{-1/12}^{1/12} \frac{\sin\big(2\pi(x-y)\big)}{\pi(x-y)}\,dy \\ &= \int_{x-1/12}^{x+1/12} \frac{\sin(2\pi u)}{\pi u}\,du \end{aligned}$$

Also notice that $|u-(k+\frac{1}{3})| < \frac{1}{6}$ if and only if $u \in \big(2\pi k + \frac{\pi}{6}, 2\pi k + \frac{5\pi}{6}\big)$ which implies

$$\sin(2\pi u) \geq \frac{1}{2}.$$

In addition $|x-(k+\frac{1}{4})| < \frac{1}{12}$, and $|u-x| < \frac{1}{12}$ implies

$$|u-(k+\tfrac{1}{4})| < \tfrac{1}{6}.$$

Therefore, we have if $x \in (k+\frac{1}{6}, k+\frac{1}{3})$ and $u \in (x-\frac{1}{12}, x+\frac{1}{12})$, then

$$\sin(2\pi u) \geq \frac{1}{2}.$$

Thus

$$\begin{aligned} \|S_1 f\|_{L^1} &\geq \sum_{k=1}^{\infty} \int_{k+1/6}^{k+1/3} \left| \int_{x-1/12}^{x+1/12} \frac{\sin(2\pi u)}{\pi u}\,du \right| dx \\ &\geq \sum_{k=1}^{\infty} \int_{k+1/6}^{k+1/3} \left(\frac{1}{2}\right)\left(\frac{1}{6}\right)\left(\frac{1}{\pi k + \pi/12}\right) dx \\ &= \sum_{k=1}^{\infty} \frac{1}{24\pi k + 2\pi} \\ &= \infty. \end{aligned}$$

□

Proposition 2.11. *Note that*

$$\mathcal{F}(f(x)e^{2\pi i a\cdot x})(\xi) = \hat{f}(\xi - a),$$

and if $f_a(x) = f(a+x)$ *then*

$$\widehat{f_a}(\xi) = e^{2\pi i \xi a}\hat{f}(\xi)$$

Proof.

$$\begin{aligned} \mathcal{F}(f(x)e^{2\pi i a\cdot x})(\xi) &= \int_{\mathbb{R}^d} f(x)e^{2\pi i a\cdot x}e^{-2\pi i \xi x}\,dx \\ &= \int_{\mathbb{R}^d} f(x)e^{-2\pi i(\xi-a)x}\,dx \\ &= \hat{f}(\xi-a). \end{aligned}$$

Similarly,

$$\widehat{f_a}(\xi) = \int_{\mathbb{R}^d} f(a+x)e^{-2\pi i \xi x}\,dx$$

$$= \int_{\mathbb{R}^d} f(v) e^{-2\pi i \xi (v-a)}\, dv$$
$$= e^{2\pi i \xi a} \hat{f}(\xi).$$

□

Definition 2.12. Let $d = 1$, for $f \in L^2(\mathbb{R})$, define the Hilbert transform

$$\mathcal{H}f(x) = \mathcal{F}^{-1}\left[-i\,\mathrm{sgn}(\xi)\hat{f}(\xi)\right](x),$$

where

$$\mathrm{sgn}(\xi) = \begin{cases} +1, & \xi > 0 \\ 0, & \xi = 0 \\ -1, & \xi < 0. \end{cases}$$

Remark 2.13. For $f \in L^2(\mathbb{R})$,

$$\|\mathcal{H}f\|_{L^2} = \|\hat{f}\|_{L^2} = \|f\|_{L^2}.$$

Proposition 2.14. *For $f \in L^2(\mathbb{R})$, let $M_a f(x) = e^{2\pi i a x} f(x)$, then*

$$\frac{i}{2}\left(M_{-1}\mathcal{H}M_1 - M_1\mathcal{H}M_{-1}\right) f = S_1 f.$$

Proof. Recall that

$$\widehat{M_a f}(\xi) = \hat{f}(\xi - a).$$

Let $A = \frac{i}{2} M_1 \mathcal{H} M_1$, then

$$\widehat{Af}(\xi) = \tfrac{i}{2}\mathcal{F}\big(\,M_{-1}(\mathcal{H}M_1 f)\,\big)(\xi)$$
$$= \tfrac{i}{2}\mathcal{F}\big(\,\mathcal{H}(M_1 f)\,\big)(\xi + 1)$$
$$= \tfrac{i}{2}\left[-i\,\mathrm{sgn}(\xi+1)\right]\widehat{M_1 f}(\xi + 1)$$
$$= \tfrac{1}{2}\,\mathrm{sgn}(\xi+1)\hat{f}(\xi).$$

Likewise, if $B = \frac{1}{2} M_{-1}\mathcal{H}M_{-1}$, then

$$\widehat{Bf}(\xi) = -\frac{i}{2}\,\mathrm{sgn}(\xi - 1)\hat{f}(\xi).$$

Thus

$$\mathcal{F}(Af + Bf)(\xi) = \frac{1}{2}\left[\mathrm{sgn}(\xi+1) - \mathrm{sgn}(\xi-1)\right]\hat{f}(\xi)$$
$$= \chi_{(-1,1)}(\xi)\hat{f}(\xi).$$

So, $\frac{i}{2}\left(M_{-1}\mathcal{H}M_1 - M_1\mathcal{H}M_{-1}\right) f = S_1 f$ for $f \in L^2(\mathbb{R})$. □

Proposition 2.15. *We have the following property in $L^p(\mathbb{R})$,*

$$\sup_{g \in L^p \cap L^1} \frac{\|S_1 g\|_{L^p}}{\|g\|_{L^p}} = \sup_{g \in \mathcal{S}} \frac{\|S_1 g\|_{L^p}}{\|g\|_{L^p}}.$$

Proof. Homework! □

Theorem 2.16. *We conclude that in $L^p(\mathbb{R})$, $1 < p < \infty$,*

$$\lim_{R \to \infty} S_R f_R = f$$

if

$$\|\mathcal{H}\|_{\mathcal{S} \to L^p} = \sup_{f \in \mathcal{S}} \frac{\|\mathcal{H}f\|_{L^p}}{\|f\|_{L^p}} < \infty.$$

Theorem 2.17 (Fefferman, 1971, *Annals of Mathematics*)**.** *In higher dimensions, where* $d > 1$

$$S_1 : L^p(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$$

is bounded if and only if $p = 2$.

2.2. **Hilbert Transform Simplification for Fourier Series.** Let $1 < p < \infty$ and $f \in L^p\big([0,1]\big)$. Recall that the N-th partial Fourier sum is

$$\begin{aligned} S_N f(x) &= \int_0^1 f(x-t)\frac{\sin\big(\pi(2N+1)t\big)}{\sin(\pi t)}\,dt \\ &= \sum_{n=-N}^{N} \hat{f}(n)e^{2\pi i n x} \\ &= \frac{1}{2}\sum_{n=-\infty}^{\infty}\big(\operatorname{sgn}(n+N)-\operatorname{sgn}(n-N)\big)\hat{f}(n)e^{2\pi i n x} \\ &\quad + \frac{1}{2}\left(\hat{f}(N)e^{2\pi i N x}+\hat{f}(-N)e^{-2\pi i N x}\right). \end{aligned}$$

Definition 2.18. Let

$$Cf(x) = \sum_{n=-\infty}^{\infty} \operatorname{sgn}(n)\hat{f}(n)e^{2\pi i n x}.$$

Note that since $\hat{f}(0) = \int_0^1 f(x)\,dx$,

$$\|Cf\|_{L^2}^2 + \left|\int_0^1 f(x)\,dx\right|^2 = \|f\|_{L^2}^2.$$

Proposition 2.19. *If the operator norm*

$$\|C\|_{L^p\to L^p} < \infty,$$

then we have

$$\sup_N \|S_N\|_{L^p\to L^p} < \infty.$$

Proof. Let

$$M_N f(x) = f(x)e^{-2\pi i N x},$$

then clearly we have

$$\widehat{M_N f}(n) = \hat{f}(n+N).$$

Thereofore,

$$\begin{aligned} \mathcal{F}(M_{-N}CM_N f)(n) &= \mathcal{F}(CM_N f)(n-N) \\ &= \operatorname{sgn}(n-N)\hat{f}(n), \end{aligned}$$

and similarly,

$$\mathcal{F}(M_N C M_{-N} f)(n) = \operatorname{sgn}(n+N)\hat{f}(n).$$

We conclude that for $f \in L^2$, by taking the inverse Fourier series,

$$\begin{aligned} &\frac{1}{2}\left[M_N C M_{-N} - M_{-N} C M_N\right] f(x) \\ &= \frac{1}{2}\sum_{n=-\infty}^{\infty}\left[\operatorname{sgn}(n+N)-\operatorname{sgn}(n-N)\right]\hat{f}(n)e^{2\pi i n x} \\ &= S_N f(x) - \frac{1}{2}\left(\hat{f}(N)e^{2\pi i N x}+\hat{f}(-N)e^{-2\pi i N x}\right). \end{aligned}$$

Hence,

$$S_N f(x) = \frac{1}{2}\left[M_N C M_{-N} - M_{-N} C M_N\right] f(x) + \frac{1}{2}\left(\hat{f}(N)e^{2\pi i N x} + \hat{f}(-N)e^{-2\pi i N x}\right).$$

Since $\|M_N\|_{L^p \to L^p} = 1$, we have

$$\|S_N\|_{L^p\to L^p} = \sup_{\substack{\|f\|_{L^p}=1,\\ f\in S}} \|S_N f\|_{L^p} \le \sup_{\substack{\|f\|_{L^p}=1,\\ f\in S}} \left(\|Cf\|_{L^p} + 1\right)$$

where

$$S = \mathrm{span}\left\{e^{2\pi i n x} : n \in \mathbb{Z}\right\}.$$

□

Question 2.20. What is $Cf(x)$ for $f \in S$?

Let

$$C_r f(x) = \sum_{n=-\infty}^{\infty} r^{|n|}\mathrm{sgn}(n)\hat{f}(n)e^{2\pi i n x}.$$

Then

$$\begin{aligned}
Cf(x) &= \lim_{r\to 1^-} C_r f(x) \\
&= \lim_{r\to 1^-}\left[\sum_{n=1}^{\infty} r^n \hat{f}(n)e^{2\pi i n x} - \sum_{n=-\infty}^{-1} r^{-n}\hat{f}(n)e^{2\pi i n x}\right] \\
&= \lim_{r\to 1^-}\int_0^1 f(t)\left[\sum_{n=1}^{\infty} r^n e^{2\pi i n(x-t)} - \sum_{n=1}^{\infty} r^n e^{-2\pi i n(x-t)}\right] dt \\
&= \lim_{r\to 1^-}\int_0^1 f(t)\left[\frac{1}{1-re^{2\pi i(x-t)}} - \frac{1}{1-re^{-2\pi i(x-t)}}\right] dt \\
&= 2i\left[\lim_{r\to 1^-}\int_0^1 f(t)\mathrm{Im}\left(\frac{1}{1-re^{2\pi i(x-t)}}\right) dt\right] \\
&= 2i\left[\lim_{r\to 1^-}\int_{-1/2}^{1/2} f(x-t)\frac{r\sin(2\pi t)}{1-2r\cos(2\pi t)+r^2}\, dt\right].
\end{aligned}$$

Since

$$\int_{-1/2}^{1/2}\left(\frac{r\sin(2\pi t)}{1-2r\cos(2\pi t)+r^2}\right) dt = 0,$$

we have

$$\begin{aligned}
Cf(x) &= 2i\lim_{r\to 1^-}\int_{-1/2}^{1/2}\big(f(x-t)-f(x)\big)\frac{r\sin(2\pi t)}{1-2r\cos(2\pi t)+r^2}dt \\
&= 2i\int_{-1/2}^{1/2}\big(f(x-t)-f(x)\big)\frac{\sin(2\pi t)}{2-2\cos(2\pi t)}dt \\
&= 2i\int_{-1/2}^{1/2}\big(f(x-t)-f(x)\big)\cot(\pi t)dt. \\
&= 2i\lim_{\epsilon\to 0^+}\int_{\epsilon<|t|<1/2}\big(f(x-t)-f(x)\big)\cot(\pi t)dt
\end{aligned}$$

$$= 2i \lim_{\epsilon \to 0^+} \int_{\epsilon<|t|<1/2} f(x-t)\cot(\pi t)dt$$
$$= 2i\mathcal{H}f(x),$$
where $\mathcal{H}f(x)$ is the Hilbert transform on the circle.

2.3. **Hilbert Transform Simplification for the Fourier Transform.**

Definition 2.21. Recall that

$$\widehat{\mathcal{H}f(x)}(\xi) = -i\,\mathrm{sgn}(\xi)\hat{f}(\xi).$$

Let

$$H_{\varepsilon,N}f(x) = \frac{1}{\pi}\int_{\varepsilon<|y|<N} \frac{f(x-y)}{y}\,dy.$$

Proposition 2.22. *As $n \to \infty$, $\exists \varepsilon(n) \to 0^+$, $N(n) \to \infty$ such that*

$$\mathcal{H}f(x) = \lim_{n\to\infty} H_{\varepsilon(n),N(n)}f(x).$$

Proof. For $f \in L^1 \cap L^2$,

$$H_{\varepsilon,N}f(x) = \frac{1}{\pi}\left(\frac{\chi_{\varepsilon<|y|<N}}{y}\right) * f.$$

So

$$\widehat{H_{\varepsilon,N}f}(\xi) = \frac{1}{\pi}\mathcal{F}\left(\frac{\chi_{\varepsilon<|y|<N}}{y}\right)(\xi)\cdot \widehat{f}(\xi),$$

where

$$\mathcal{F}\left(\frac{\chi_{\varepsilon<|y|<N}}{y}\right) \in L^1 \cap L^2.$$

We compute

$$\frac{1}{\pi}\mathcal{F}\left(\frac{\chi_{\varepsilon<|y|<N}}{y}\right)(\xi) = \frac{1}{\pi}\int_{\varepsilon<|y|<N} \frac{e^{-2\pi i\xi y}}{y}\,dy = -\frac{1}{\pi}\int_{\varepsilon<|y|<N} \frac{e^{2\pi i\xi y}}{y}\,dy.$$

Therefore,

$$\frac{1}{\pi}\mathcal{F}\left(\frac{\chi_{\varepsilon<|y|<N}}{y}\right)(\xi) = -\frac{1}{2\pi}\int_{\varepsilon<|y|<N} \frac{e^{2\pi i\xi y} - e^{-2\pi i\xi y}}{y}\,dy.$$
$$= -\frac{i}{\pi}\int_{\varepsilon<|y|<N} \frac{\sin(2\pi\xi y)}{y}\,dy$$
$$= -\frac{i}{\pi}\left(\int_{-N}^{-\varepsilon} \frac{\sin(2\pi\xi y)}{y}\,dy + \int_{\varepsilon}^{N} \frac{\sin(2\pi\xi y)}{y}\,dy\right).$$

Case I: $x > 0$, let $u = 2\pi y$, then

$$\frac{1}{\pi}\mathcal{F}\left(\frac{\chi_{\varepsilon<|y|<N}}{y}\right)(\xi) = \frac{-i}{\pi}\left(\int_{-2\pi N\xi}^{-2\pi\varepsilon\xi} \frac{\sin u}{u}\,du + \int_{2\pi\varepsilon\xi}^{2\pi N\xi} \frac{\sin u}{u}\,dy\right)$$
$$= -\frac{2i}{\pi}\int_{2\pi\varepsilon|\xi|}^{2\pi N|\xi|} \frac{\sin y}{y}\,dy.$$

Case II: $x > 0$, then $\frac{1}{\pi}\mathcal{F}\left(\frac{\chi_{\varepsilon<|y|<N}}{y}\right)(\xi) = 0$.

Case III: $x < 0$, let $u = -2\pi y$, then

$$\frac{1}{\pi}\mathcal{F}\left(\frac{\chi_{\varepsilon<|y|<N}}{y}\right)(\xi) = \frac{i}{\pi}\left(\int_{-N(-2\pi\xi)}^{-\varepsilon(-2\pi\xi)} \frac{\sin u}{u}\,du + \int_{\varepsilon(-2\pi\xi)}^{N(-2\pi\xi)} \frac{\sin u}{u}\,du\right).$$

Therefore,

$$\frac{1}{\pi}\mathcal{F}\left(\frac{\chi_{\varepsilon<|y|<N}}{y}\right)(\xi) = -\frac{2i}{\pi}\mathrm{sgn}(x)\int_{2\pi\varepsilon|\xi|}^{2\pi N|\xi|} \frac{\sin y}{y}\,dy.$$

So

$$\widehat{H_{\varepsilon,N}f}(\xi) = -\frac{2i}{\pi}\mathrm{sgn}(x)\int_{2\pi\varepsilon|\xi|}^{2\pi N|\xi|}\frac{\sin y}{y}\,dy\cdot\widehat{f}(\xi).$$

Then, by the classical integral $\int_0^\infty \frac{\sin u}{u}du = \frac{\pi}{2}$,

$$\|H_{\varepsilon,N}f - \mathcal{H}f\|_{L^2}^2 = \|\widehat{H_{\varepsilon,N}f} - \widehat{\mathcal{H}f}\|_{L^2}^2$$

$$= \int_{\mathbb{R}}\left|\left(\left(\frac{2}{\pi}(-i)\,\mathrm{sgn}(\xi)\int_{2\pi\varepsilon|\xi|}^{2\pi N|\xi|}\frac{\sin y}{y}\,dy - i\mathrm{sgn}(\xi)\right)\widehat{f}(\xi)\right)\right|^2 d\xi$$

$\to 0$ as $\varepsilon\to 0^+$, $N\to\infty$. So when $\xi\neq 0$,

$$\lim_{\varepsilon\to 0^+, N\to\infty}\left(\frac{2}{\pi}\int_{2\pi\varepsilon|\xi|}^{2\pi N|\xi|}\frac{\sin y}{y}\,dy\right) = 1.$$

□

Remark 2.23. Now,

$$H_{\varepsilon,N}f(x) = \int_{\varepsilon<|y|<N}\frac{f(x-y)}{y}\,dy$$

$$= \int_{\varepsilon<|y|<1}\frac{f(x-y)}{y}\,dy + \int_{1\leq|y|<N}\frac{f(x-y)}{y}\,dy$$

$$= \int_{\varepsilon<|y|<1}\frac{f(x-y)-f(x)}{y}\,dy + \underbrace{\int_{\varepsilon<|y|<1}\frac{f(x)}{y}\,dy}_{=0 \text{ because } \left(\int_{-1}^{-\varepsilon}+\int_{\varepsilon}^{1}\right)\frac{1}{y}dt=0} + \int_{1\leq|y|\leq N}\frac{f(x-y)}{y}\,dy$$

So if $f\in\mathcal{S}(\mathbb{R})$,

$$\mathcal{H}f(x) = \frac{1}{\pi}\int_{|y|<1}\frac{f(x-y)-f(x)}{y}\,dy + \frac{1}{\pi}\int_{|y|\geq 1}\frac{f(x-y)}{y}\,dy$$

$$= \frac{1}{\pi}\lim_{\varepsilon\to 0^+}\int_{\varepsilon<|y|<1}\frac{f(x-y)-f(x)}{y}\,dy + \frac{1}{\pi}\int_{|y|\geq 1}\frac{f(x-y)}{y}\,dy$$

Alternatively,

$$\mathcal{H}f(x) = \lim_{\varepsilon\to 0^+}\frac{1}{\pi}\int_{|y|>\varepsilon}\frac{f(x-y)}{y}\,dy.$$

2.4. Marcinkiewicz Interpolation Theorem.

Remark 2.24. Easy homework:

$$m\left(\{x \in \mathbb{R}^n : |f(x)| > \lambda\}\right) \le \frac{\|f\|_{L^q}^q}{\lambda^q}$$

Thus, if $T : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is bounded,

$$m\left(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}\right) \le \frac{\|Tf\|_{L^q}^q}{\lambda^q} \le \frac{C^q\|f\|_{L^q}^q}{\lambda^q}.$$

Definition 2.25. Let M denote the space of measurable functions, the operator $T : L^p \to M$ is said to be weak type (p,q), $q < \infty$, if

$$m\left(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}\right) \le \frac{C^q\|f\|_{L^p}^q}{\lambda^q}.$$

Remark 2.26. It is easy to check that

$$\int_{\mathbb{R}^n} |f(x)|^q dx = q\int_0^\infty \lambda^{q-1} m\left(\{x \in \mathbb{R}^n : |f(x)| > \lambda\}\right) d\lambda$$

where

$$d_f(\lambda) = m\left(\{x \in \mathbb{R}^n : |f(x)| > \lambda\}\right)$$

is the distribution function.

Definition 2.27. Let $M' \subseteq M$ be a vector space. A mapping $T : M' \to M$ is sublinear if

(1) $|T(f_0 + f_1)(x)| \le |Tf_0(x)| + |Tf_1(x)|$,

(2) $|T(\lambda f)(x)| \le |\lambda||Tf(x)|$.

Theorem 2.28 (Marcinkiewicz Interpolation Theorem)**.** *Let $1 \le p_0 < p_1 \le \infty$ and suppose $T : L^{p_0} + L^{p_1} \to M$ is sublinear. If T is weak type (p_0, p_0) and weak type (p_1, p_1), then for $p_0 < p < p_1$, we have*

$$\|Tf\|_{L^p} \le 2p^{\frac{1}{p}}\left[\frac{1}{p-p_0} + \frac{1}{p_1 - p}\right]^{\frac{1}{p}} C_0^\theta C_1^{1-\theta}\|f\|_{L^p},$$

where

$$\frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}, \quad 0 < \theta < 1.$$

Proof. Case $p_1 = \infty$: homework.

Case $p_1 < \infty$: Let $f \in L^p$. Define

$$f_1 = f \cdot \chi_{\{x:|f(x)|<C\lambda\}},$$
$$f_0 = f \cdot \chi_{\{x:|f(x)|\ge C\lambda\}},$$

so $f = f_0 + f_1 \in L^{p_0} + L^{p_1}$. Note that

$$|Tf(x)| \le |Tf_0(x)| + |Tf_1(x)|,$$

and

$$\{x : |Tf(x)| > \lambda\} \subseteq \left\{x : |Tf_0(x)| > \frac{\lambda}{2}\right\} \cup \left\{x : |Tf_1(x)| > \frac{\lambda}{2}\right\},$$

therefore,

$$d_{Tf}(\lambda) \le d_{Tf_0}\left(\frac{\lambda}{2}\right) + d_{Tf_1}\left(\frac{\lambda}{2}\right).$$

Thus,

$$\|Tf\|_{L^p}^p = p\int_0^\infty \lambda^{p-1} d_{Tf}(\lambda)\, d\lambda$$
$$\le p\left[\int_0^\infty \lambda^{p-1} d_{Tf_0}\left(\frac{\lambda}{2}\right) d\lambda + \int_0^\infty \lambda^{p-1} d_{Tf_1}\left(\frac{\lambda}{2}\right) d\lambda\right].$$

Furthermore,

$$\begin{aligned} d_{Tf_1}\left(\frac{\lambda}{2}\right) &= m\left(\{x : |Tf_1(x)| > \lambda/2\}\right) \\ &\leq {C_1}^{p_1}\left(\frac{2}{\lambda}\right)^{p_1} \|f_1\|_{L^{p_1}}^{p_1} \\ &= C_1\left(\frac{2}{\lambda}\right)^{p_1} \int_{\mathbb{R}^n} |f(x)|^{p_1}\chi_{\{x:|f(x)|<C\lambda\}}dx. \end{aligned}$$

Likewise, we have

$$d_{Tf_0}\left(\frac{\lambda}{2}\right) \leq C_0\left(\frac{2}{\lambda}\right)^{p_0} \int_{\mathbb{R}^n} |f(x)|^{p_0}\chi_{\{x:|f(x)|\geq C\lambda\}}dx$$

Thus,

$$\begin{aligned} \|Tf\|_{L^p}^p \leq p\Bigg[&(2C_0)^{p_0}\int_0^\infty \left(\int_{\mathbb{R}^n} |f(x)|^{p_0}\lambda^{p-1-p_0}\chi_{\{x:|f(x)|\geq C\lambda\}}dx\right)d\lambda \\ &+(2C_1)^{p_1}\int_0^\infty \left(\int_{\mathbb{R}^n} |f(x)|^{p_1}\lambda^{p-1-p_1}\chi_{\{x:|f(x)|<C\lambda\}}dx\right)d\lambda\Bigg] \\ = p\Bigg[&(2C_0)^{p_0}\int_{\mathbb{R}^n} |f(x)|^{p_0}\left(\int_0^\infty \lambda^{p-1-p_0}\chi_{\left(0,\frac{|f(x)|}{C}\right]}(\lambda)d\lambda\right)dx \\ &+(2C_1)^{p_1}\int_{\mathbb{R}^n} |f(x)|^{p_1}\left(\int_0^\infty \lambda^{p-1-p_1}\chi_{\left(\frac{|f(x)|}{C},\infty\right)}(\lambda)d\lambda\right)dx\Bigg] \\ = p\Bigg[&(2C_0)^{p_0}\int_{\mathbb{R}^n} |f(x)|^{p_0}\left(\int_0^{\frac{|f(x)|}{C}} \lambda^{p-1-p_0}d\lambda\right)dx \\ &+(2C_1)^{p_1}\int_{\mathbb{R}^n} |f(x)|^{p_1}\left(\int_{\frac{|f(x)|}{C}}^\infty \lambda^{p-1-p_1}d\lambda\right)dx\Bigg] \\ = p\Bigg[&(2C_0)^{p_0}\int_{\mathbb{R}^n} |f(x)|^{p_0}\left(\frac{|f(x)|^{p-p_0}}{(p-p_0)C^{p-p_0}}\right)dx \\ &+(2C_1)^{p_1}\int_{\mathbb{R}^n} |f(x)|^{p_1}\left(\frac{|f(x)|^{p-p_1}}{(p_1-p)C^{p-p_1}}\right)dx\Bigg] \\ = p&\left[\frac{(2C_0)^{p_0}}{(p-p_0)C^{p-p_0}} + \frac{(2C_1)^{p_1}}{(p_1-p)C^{p-p_1}}\right]\|f\|_{L^p}^p \\ = \frac{p}{C^p}&\left[\frac{(2C_0C)^{p_0}}{(p-p_0)} + \frac{(2C_1C)^{p_1}}{(p_1-p)}\right]\|f\|_{L^p}^p. \end{aligned}$$

Pick C where $(2C_0C)^{p_0} = (2C_1C)^{p_1}$, then

$$\frac{p}{C^p}\left[\frac{(2C_0C)^{p_0}}{(p-p_0)} + \frac{(2C_1C)^{p_1}}{(p_1-p)}\right] = \frac{p(2C_0C)^{p_0}}{C^p}\left[\frac{1}{p-p_0} + \frac{1}{p_1-p}\right].$$

Therefore,

$$\|Tf\|_{L^p} \leq p^{1/p}\frac{(2C_0C)^{p_0/p}}{C}\left[\frac{1}{p-p_0} + \frac{1}{p_1-p}\right]^{1/p}\|f\|_{L^p}.$$

Let $\frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}$, then note that $C = \frac{(2C_0C)^{p_0/p}}{2C_1}$ yields

$$\begin{aligned} \frac{(2C_0C)^{p_0/p}}{C} &= 2C_1(2C_0C)^{\frac{p_0}{p}-\frac{p_0}{p_1}} \\ &= 2C_1(2C_0C)^{\theta\left[1-\frac{p_0}{p_1}\right]}. \end{aligned}$$

Since $2C_1C = (2C_0C)^{p_0/p_1}$,

$$\frac{C_1}{C_0} = \frac{2C_1C}{2C_0C} = (2C_0C)^{\frac{p_0}{p_1}-1},$$

$$\left(\frac{C_0}{C_1}\right)^{\theta} = (2C_0C)^{\left[1-\frac{p_0}{p_1}\right]\theta}.$$

So

$$\frac{(2C_0C)^{\frac{p_0}{p}}}{C} = 2C_1\left(\frac{C_0}{C_1}\right)^{\theta} = 2C_1^{1-\theta}C_0^{\theta}.$$

□

3. Dyadic Intervals

3.1. What are Dyadic Intervals?

Definition 3.1. We define the dyadic intervals

$$\mathcal{D} = \left\{ \left[\frac{l-1}{2^k}, \frac{l}{2^k} \right) : (l,k) \in \mathbb{Z} \times \mathbb{Z} \right\},$$

and

$$\mathcal{D}_k = \left\{ I = \left[\frac{l-1}{2^k}, \frac{l}{2^k} \right) : l \in \mathbb{Z} \right\}.$$

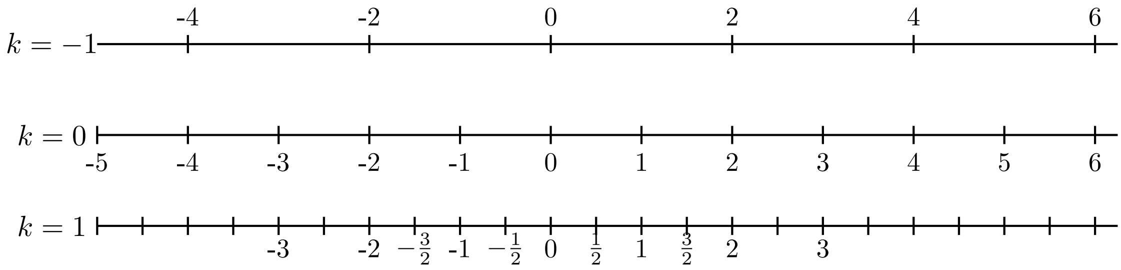

Figure 5. Dyadic Intervals with $l \in \mathbb{Z}$

Proposition 3.2. *Notice that each interval is contained in uniquely one upper-level interval.*

1. $\mathbb{R} = \bigsqcup_{I \in \mathcal{D}_k} I$.
2. $\forall I \in \mathcal{D}$, $\exists \widetilde{I} \in \mathcal{D}$ *such that* $I \subseteq \widetilde{I}$, $\ell(\widetilde{I}) = 2\ell(I)$.
3. *For* $I, J \in \mathcal{D}$, *we have* $I \cap J \in \{\varnothing, I, J\}$ *and* $I \subsetneq J$ *if and only if*
$$J = \widetilde{I}^{(k)} \text{ for some } k \in \mathbb{N},$$
where $\widetilde{I}^{(k)}$ *is the* k*-th generational parent, where*
$$\widetilde{I}^{(0)} = I, \quad \widetilde{I}^{(k)} = \widetilde{\left(\widetilde{I}^{(k-1)}\right)}, \quad \widetilde{I}^{(1)} = \widetilde{\widetilde{I}^{(0)}} = \widetilde{I}.$$
4. *If* $I = (a,b), \ell(I) = b - a$, *and*
$$2^{-k-1} \le \ell(I) < 2^{-k},$$
then
$$\#\{J \in \mathcal{D}_k : I \cap J \neq \varnothing\} \le 2.$$

Proof. (1): Obvious.

(2): Let $I = \left[\frac{l-1}{2^k}, \frac{l}{2^k}\right)$ and let l be even. Then

$$I = \left[\frac{l/2 - 1/2}{2^{k-1}}, \frac{l/2}{2^{k-1}} \right) \subseteq \left[\frac{l/2 - 1}{2^{k-1}}, \frac{l/2}{2^{k-1}} \right) = \widetilde{I}.$$

and

$$\ell(\widetilde{I}) = \frac{1}{2^{k-1}} = \frac{2}{2^k} = 2\ell(I)$$

If l is odd, then

$$I = \left[\frac{(l-1)/2}{2^{k-1}}, \frac{l/2}{2^{k-1}} \right) \subseteq \left[\frac{(l-1)/2}{2^{k-1}}, \frac{(l+1)/2}{2^{k-1}} \right) = \widetilde{I}.$$

(3): Homework.

(4): Let

$$\{J \in \mathcal{D}_k : J \cap I \neq \varnothing\} = \{J_i\}_{i=1}^M.$$

Let $J_i = [a_i, b_i)$, $c_i \in J_i \cap I$, WLOG assume

$$a_1 \le c_1 < b_1 \le a_2 < b_2 \le \cdots \le a_{M-1} < b_{M-1} \le a_M \le c_M < b_M.$$

So

$$\begin{aligned} b - a &\ge c_M - c_1 > b_M - a_2 \\ &\ge \sum_{i=2}^{M-1} (b_i - a_i) \\ &= (M-2)2^{-k}. \end{aligned}$$

Thus

$$2^{-k} > b - a \ge (M-2)2^{-k},$$

so $3 > M$. □

Remark 3.3. (4) says if I is an interval, $\exists \{J_i\}_{i=1}^2 \subseteq \mathcal{D}$ such that $I \subseteq J_1 \sqcup J_2$ and if $\ell = \ell(J_1) = \ell(J_2)$ then

$$\frac{\ell}{2} \le \ell(I) < \ell.$$

3.2. **The Hardy-Littlewood Maximal Function.**

Definition 3.4 (Hardy-Littlewood Maximal Function)**.** Let $f \in L^1(\mathbb{R})$. Let $Mf : \mathbb{R} \to [0, \infty)$,

$$Mf(x) = \sup_{r>0} \frac{1}{2r} \int_{x-r}^{x+r} |f(y)| \, dy.$$

Remark 3.5. If $f \in L^\infty(\mathbb{R})$,

$$\frac{1}{2r} \int_{x-r}^{x+r} |f(y)| \, dy \le \frac{2r}{2r} \|f\|_{L^\infty} = \|f\|_{L^\infty}.$$

Therefore,

$$\|Mf\|_{L^\infty} \le \|f\|_{L^\infty}.$$

Theorem 3.6. *M is weak type $(1,1)$. $Mf \notin L^1$ if $f \neq 0$ a.e.*

Proof. Let $\lambda > 0$. Step (1): we claim

$$\begin{aligned} \{x \in \mathbb{R} : Mf(x) > \lambda\} &\subseteq \bigcup_{Q \in \mathcal{D}} \left\{ 2Q : Q \in \mathcal{D}, \frac{1}{\ell(Q)} \int_Q |f(y)| \, dy > \frac{\lambda}{4} \right\} \\ &= \bigcup_{Q \in \mathcal{D}} \mathscr{C}. \end{aligned}$$

Assume that

$$x \notin \bigcup_{Q \in \mathcal{D}} \mathscr{C}.$$

Let $I = (x - r, x + r)$, let $\{J_i\}_{i=1}^2$ as before, i.e., $I \subseteq J_1 \sqcup J_2$. We claim that $J_i \notin \mathscr{C}$. If not,

$$J_i \in \mathscr{C}, \text{ and let } u \in I \cap J_i.$$

then

$$|x - c_i| \le |x - u| + |u - c_i|$$

where c_i is the center of J_i. Then,

$$|x - c_i| \le \frac{\ell(I)}{2} + \frac{\ell(J_i)}{2} < \ell(J_i).$$

Thus, $x \in 2J_i$ and $J_i \in \mathscr{C}$, so $x \in \mathscr{C}$, contradiction. Thus

$$\begin{aligned}\frac{1}{\ell(I)}\int_I |f(y)|\,dy \\ \leq \sum_{i=1}^{2} \frac{1}{\ell(J_i)}\int_{J_i} |f(y)|\,dy \\ \leq \sum_{i=1}^{2} 2\left(\frac{\lambda}{4}\right) = \lambda.\end{aligned}$$

Thus

$$\frac{1}{2r}\int_{x-r}^{x+r} |f(y)|\,dy \leq \lambda \text{ for } x \notin \mathscr{C}.$$

Step(2): We claim that

$$m\left(\bigcup_{Q\in\mathcal{D}}\left\{2Q : Q \in \mathcal{D}, \frac{1}{\ell(Q)}\int_Q |f(y)|\,dy > \frac{\lambda}{4}\right\}\right) \leq \frac{8}{\lambda}\|f\|_{L^1}.$$

Let $Q \in \mathscr{C}$, and

$$Q = \widetilde{Q}^{(0)} \subseteq \widetilde{Q}^{(1)} \subseteq \widetilde{Q}^{(2)} \cdots$$

where $\ell(\widetilde{Q}^{(k)}) = 2^k\ell(Q)$. Therefore,

$$\frac{1}{\ell(\widetilde{Q}^{(k)})}\int_{\widetilde{Q}^{(k)}} |f(y)|\,dy \leq \frac{\|f\|_{L^1}}{2^k\ell(Q)} \to 0 \text{ as } k \to \infty.$$

Thus we can conclude that

(A) $\widetilde{Q}^{(0)} = Q \in \mathscr{C}$.

(B) $\widetilde{Q}^{(k)} \notin \mathscr{C}$ for k large.

For every $Q \in \mathscr{C}$, $\exists M \in \mathbb{N}$ such that $\widetilde{Q}^{(M)} \in \mathscr{C}$ and $\widetilde{Q}^{(n)} \notin \mathscr{C}$ for $n > M$. Define the **Calderón-Zygmund intervals** of height λ:

$$\begin{aligned}\{Q_j\} &= \left\{P \in \mathscr{C} : \widetilde{P}^{(k)} \notin \mathscr{C} \text{ for any } k \in \mathbb{N}\right\} \\ &= \left\{P \in \mathscr{C} : P' \supsetneq P \implies P' \notin \mathscr{C}\right\}.\end{aligned}$$

Claim that $\{Q_j\}$ are disjoint. Why? If $P, P' \in \{Q_j\}$ then $P \nsubseteq P'$ or $P' \nsubseteq P$, impossible. Notice that

$$\frac{1}{\ell(Q_j)}\int_{Q_j} |f(y)|\,dy > \frac{\lambda}{4}$$

implies

$$\frac{4}{\lambda}\int_{Q_j} |f(y)|\,dy > \ell(Q_j).$$

Therefore,

$$\begin{aligned}\sum_j^{\infty} \ell(Q_j) &\leq \frac{4}{\lambda}\sum_j^{\infty}\int_{Q_j} |f(y)|\,dy \\ &\leq \frac{4}{\lambda}\int_{\mathbb{R}} |f(y)|\,dy \\ &= \frac{4\|f\|_{L^1}}{\lambda}.\end{aligned}$$

Finally,

$$m\left(\bigcup_{Q\in\mathscr{C}} 2Q\right) \leq m\left(\bigcup_j 2Q_j\right) \leq \sum_j m(2Q_j)$$

$$\leq 2\sum_j \ell(Q_j) \leq \frac{8}{\lambda}\|f\|_{L^1}.$$

□

Corollary 3.7. *$M : L^p \to L^p$ is bounded for $1 < p \leq \infty$.*

3.3. **Calderón-Zygmund decomposition.**

Theorem 3.8 (Lebesgue Differentiation Theorem)**.** *Let $f \in L^1$. For a.e. $x \in \mathbb{R}$ we have*

$$\lim_{r\to 0^+} \frac{1}{2r}\int_{x-r}^{x+r} |f(y) - f(x)|\, dy = 0.$$

In particular, for a.e. $x \in \mathbb{R}$,

$$f(x) = \lim_{r\to 0^+} \frac{1}{2r}\int_{x-r}^{x+r} f(y)\, dy.$$

Proof. Notice that

$$\left|f(x) - \frac{1}{2r}\int_{x-r}^{x+r} f(y)\, dy\right| = \left|\frac{1}{2r}\int_{x-r}^{x+r} (f(x) - f(y))\, dy\right|$$
$$\leq \frac{1}{2r}\int_{x-r}^{x+r} |f(x) - f(y)|\, dy,$$

which we will prove $\to 0$ as $r \to 0^+$. Let

$$T_r f(x) = \frac{1}{2r}\int_{x-r}^{x+r} |f(x) - f(y)|\, dy,$$

and

$$Tf(x) = \limsup_{r\to 0^+} \frac{1}{2r}\int_{x-r}^{x+r} |f(x) - f(y)|\, dy.$$

We claim that $m(\{x \in \mathbb{R} : Tf(x) > \lambda\}) = 0$, which is equivalent to showing

$$m(\{x \in \mathbb{R} : Tf(x) > \lambda\}) = 0$$

for all $\lambda > 0$. Pick $g \in C(\mathbb{R})$ such that $\|f - g\|_{L^1} < \varepsilon$. Let $h = f - g$. Then

$$T_r f \leq T_r g + T_r h.$$

Notice that

$$T_r h(x) \leq \frac{1}{2r}\int_{x-r}^{x+r} |h(y)|\, dy + |h(x)|.$$
$$\leq Mh(x) + |h(x)|.$$

where $Mh(x)$ is the Hardy-Littlewood maximal function. So

$$Tf(x) \leq \limsup_{r\to 0^+} (T_r g(x) + Mh(x) + |h(x)|).$$
$$\leq Mh(x) + |h(x)|.$$

Clearly we have

$$\{x : Tf(x) > \lambda\} \subseteq \left\{x : Mh(x) > \frac{\lambda}{2}\right\} \cup \left\{x : |h(x)| > \frac{\lambda}{2}\right\}.$$

Therefore,

$$m(\{x : Tf(x) > \lambda\}) \leq m\left(\left\{x : Mh(x) > \frac{\lambda}{2}\right\}\right) + m\left(\left\{x : |h(x)| > \frac{\lambda}{2}\right\}\right).$$
$$\leq C_1\frac{2}{\lambda}\|h\|_{L^1} + \frac{2}{\lambda}\|h\|_{L^1},$$

where C_1 is the type $(1,1)$ bound for the maximal function, and the latter term comes from Chebyshev's inequality. Recall $\|h\|_{L^1} = \|f-g\|_{L^1} < \varepsilon$, so

$$m(\{x : Tf(x) > \lambda\}) \leq \frac{2C_1 + 2}{\lambda}\varepsilon.$$

As ϵ is arbitrary,

$$m(\{x : Tf(x) > \lambda\}) = 0.$$

□

Corollary 3.9. *Let $f \in L^1$, $F(x) = \int_a^x f(y)\,dy$. Then for a.e. $x \in \mathbb{R}$,*

$$F'(x) = f(x).$$

Proof. Notice that

$$\begin{aligned}
&\left|\frac{F(x+r) - F(x)}{r} - f(x)\right| \\
&= \left|\frac{1}{r}\int_a^{x+r} f(y)\,dy - \frac{1}{r}\int_a^x f(y)\,dy - f(x)\right| \\
&= \left|\frac{1}{r}\int_x^{x+r} f(y)\,dy - f(x)\right| \\
&\leq \frac{1}{r}\int_x^{x+r} |f(y) - f(x)|\,dy \\
&< 2\left[\frac{1}{2r}\int_{x-r}^{x+r} |f(y) - f(x)|\,dy\right]
\end{aligned}$$

which $\to 0$ as $r \to 0^+$ by the Lebesgue differentiation theorem. □

Definition 3.10. Define the Calderón-Zygmund intervals of height λ:

$$\begin{aligned}
\{Q_j\} &= \left\{P \in \mathscr{C} : \widetilde{P}^{(k)} \notin \mathscr{C} \text{ for any } k \in \mathbb{N}\right\} \\
&= \left\{P \in \mathscr{C} : P' \supsetneq P \implies P' \notin \mathscr{C}\right\}.
\end{aligned}$$

where

$$\mathscr{C} = \left\{P \in \mathcal{D} : \frac{1}{\ell(P)}\int_P f(y)\,dy > \lambda\right\}.$$

Lemma 3.11. *Let $\{Q_j\}$ be Calderón-Zygmund intervals of height λ for $f \geq 0$. Then:*

1. *$f(x) \leq \lambda$ for a.e. $x \notin \bigsqcup_j^\infty Q_j$.*
2. *$m\left(\bigsqcup_j^\infty Q_j\right) = \sum_j^\infty \ell(Q_j) \leq \frac{\|f\|_{L^1}}{\lambda}$.*
3. *$\lambda < \frac{1}{\ell(Q_j)}\int_{Q_j} f(y)\,dy \leq 2\lambda$.*

Proof. (2): Done already!

(1): Pick $I_k \in \mathcal{D}_k$ such that $x \in I_k$, so $\{x\} = \bigcap_{k=-\infty}^\infty I_k$. By the Lebesgue Differentiation Theorem,

$$f(x) = \lim_{k\to\infty} \frac{1}{\ell(I_k)}\int_{I_k} f(y)\,dy.$$

Claim that if $x \notin \bigsqcup_j Q_j$, then for all $k \in \mathbb{N}$,

$$\frac{1}{\ell(I_k)}\int_{I_k} f(y)\,dy \leq \lambda.$$

If not, $I_k \in \mathscr{C}$ implies $I_k \subseteq Q_{j'}$ for some j'. Thus $x \in I_k \subseteq \bigsqcup_j Q_j$, contradiction. Hence,

$$f(x) = \lim_{k\to\infty} \frac{1}{\ell(I_k)} \int_{I_k} f(y)\, dy \le \lambda.$$

(3): By definition,

$$\frac{1}{\ell(Q_j)} \int_{Q_j} f(y)\, dy > \lambda.$$

Also by definition $\widetilde{Q_j} \notin \mathscr{C}$, thus

$$\frac{1}{\ell(Q_j)} \int_{Q_j} f(y)\, dy \le \frac{2}{\ell(\widetilde{Q_j})} \int_{\widetilde{Q_j}} f(y)\, dy \le 2\lambda.$$

□

Theorem 3.12 (Calderón-Zygmund decomposition of $f \ge 0$ at height λ)**.** *Let $\lambda > 0$ and $f \in L^1$. If $\{Q_j\}$ are Calderón-Zygmund intervals; then $(1) - (3)$ previously are true.*

Further, define

$$g(x) = \begin{cases} f(x) & x \notin \bigsqcup_j Q_j, \\ \frac{1}{\ell(Q_j)} \int_{Q_j} f(y)\, dy & x \in Q_j, \end{cases}$$

where $b(x) = f(x) - g(x)$ is the "bad" function. Then $f(x) = g(x) + b(x)$ satisfies:

- (A) $\|g\|_{L^\infty(\mathbb{R})} \le 2\lambda$ *and* $\|g\|_{L^1(\mathbb{R})} \le \|f\|_{L^1(\mathbb{R})}$.
- (B) $b(x) = \sum_j b_j(x)$, *where*
  - (B1) $b_j(x) = 0$ *for* $x \notin Q_j$,
  - (B2) $\int_{Q_j} b_j(y)\, dy = 0$,
  - (B3) $\int_{Q_j} |b_j(y)|\, dy \le 4\lambda\ell(Q_j)$.

Proof. (A): If $x \notin \bigsqcup_j Q_j$, then $g(x) = f(x) \le \lambda$. If $x \in Q_j$, then

$$g(x) = \frac{1}{\ell(Q_j)} \int_{Q_j} f(y)\, dy \le 2\lambda.$$

Also,

$$\begin{aligned} \int_{\mathbb{R}} g(y)dy &= \int_{\bigsqcup_j Q_j} g(y)dy + \int_{\mathbb{R}\setminus \bigsqcup_j Q_j} g(y)dy \\ &= \sum_j \int_{Q_j} f(y)dy + \int_{\mathbb{R}\setminus \bigsqcup_j Q_j} f(y)dy \\ &= \|f\|_{L^1(\mathbb{R})}. \end{aligned}$$

(B): For $x \notin \bigsqcup_j Q_j$,

$$b(x) = f(x) - g(x) = f(x) - f(x) = 0.$$

For $x \in Q_j$,

$$b(x) = f(x) - \frac{1}{\ell(Q_j)} \int_{Q_j} f(y)\, dy.$$

So

$$b(x) = \sum_j \left(f(x) - \fint_{Q_j} f(y)\, dy \right) \chi_{Q_j}(x) = \sum_j b_j(x),$$

where

$$\fint_{Q_j} f = \frac{1}{\ell Q_j} \int_{Q_j} f(y)\, dy.$$

Then

$$\int_{Q_j} b_j(y)\,dy = \int_{Q_j} f(y)\,dy - \int_{Q_j} f(y)\,dy = 0.$$

Finally,

$$\begin{aligned}\int_{Q_j} |b_j(y)|\,dy &\leq \int_{Q_j} \left(f(y) + \fint_{Q_j} f \right) dy \\ &= 2\int_{Q_j} f(y)\,dy \\ &\leq 2\ell(Q_j)\left(\frac{1}{\ell(Q_j)} \int_{Q_j} f(y)\,dy \right) \leq \big(2\ell(Q_j) \big)(2\lambda).\end{aligned}$$

□

3.4. **Weak type estimates for the Hilbert transform.**

Lemma 3.13. *Let $f \in L^2$, $f = 0$ a.e. on $\mathbb{R} \setminus I$, where I is compact. Then*

$$\mathcal{H}f(x) = \frac{1}{\pi} \int_I \frac{f(y)}{x-y}\,dy$$

for a.e. $x \in \mathbb{R}/I$.

Theorem 3.14. *If $f \in L^1 \cap L^2$, then*

$$m\left(\{x : |\mathcal{H}f(x)| > \lambda\}\right) \leq \frac{C\|f\|_1}{\lambda}.$$

Proof. WLOG $f \geq 0$, let $f = g + b$ be the Calderón–Zygmund decomposition at height $\lambda > 0$. Then clearly

$$m\left(\{x : |\mathcal{H}f(x)| > \lambda\}\right) \leq \underbrace{m\left(\left\{x : |\mathcal{H}g(x)| > \frac{\lambda}{2}\right\}\right)}_{(1)} + \underbrace{m\left(\left\{x : |\mathcal{H}b(x)| > \frac{\lambda}{2}\right\}\right)}_{(2)}.$$

(1): Notice that

$$\begin{aligned}(1) &\leq \left(\frac{2}{\lambda}\right)^2 \int_{\mathbb{R}} |\mathcal{H}g(x)|^2\,dx \\ &= \left(\frac{2}{\lambda}\right)^2 \int_{\mathbb{R}} |g(x)|^2\,dx.\end{aligned}$$

Since $\|g\|_\infty \leq 2\lambda$, and $\|g\|_2 \leq \|f\|_2$,

$$\begin{aligned}(1) &\leq \left(\frac{2}{\lambda}\right)^2 (2\lambda) \int_{\mathbb{R}} |g(x)|dx \\ &\leq \frac{8}{\lambda}\|f\|_1.\end{aligned}$$

(2): Recall that

$$b(x) = \sum_j \left(f(x) - \fint_{Q_j} f(y)\,dy \right) \chi_{Q_j}(x),$$

we have

$$\begin{aligned}\|b\|_{L^2}^2 &= \sum_j \int_{Q_j} \left(f(x) - \fint_{Q_j} f(y)\,dy \right)^2 dx \\ &\leq 2\sum_j \left[\int_{Q_j} \big(f(x) \big)^2 + \ell(Q_j) \left(\fint_{Q_j} f(y)\,dy \right)^2 dx \right]\end{aligned}$$

$$\le 4\sum_j \int_{Q_j} \big(f(x)\big)^2 dx$$
$$\le 4\int_{\mathbb{R}} |f(x)|^2 dx < \infty.$$

Therefore, $b \in L^2$ and $b = \sum_j b_j \in L^2$. Then $\mathcal{H}b(x) = \sum_j \mathcal{H}b_j(x)$ in L^2.

$$|\mathcal{H}b(x)| \le \sum_j |\mathcal{H}b_j(x)| \quad \text{for a.e. } x \in \mathbb{R}.$$

Define

$$\Omega = \bigsqcup_j Q_j, \quad \Omega^* = \bigcup_j 2Q_j.$$

Then

$$m(\Omega^*) \le \sum_j m(2Q_j)$$
$$= 2\sum_j m(Q_j)$$
$$\le \frac{2\|f\|_{L^1}}{\lambda}.$$

Therefore,

$$(2) \le m(\Omega^*) + m\left(\left\{x \in \mathbb{R}\setminus\Omega^* : |\mathcal{H}b(x)| > \frac{\lambda}{2}\right\}\right)$$
$$\le \frac{2\|f\|_1}{\lambda} + \frac{2}{\lambda}\int_{\mathbb{R}\setminus\Omega^*} |\mathcal{H}b(x)| dx.$$

We claim that $\frac{2}{\lambda}\int_{\mathbb{R}\setminus\Omega^*} |\mathcal{H}b(x)| dx \le \frac{C}{\lambda}\|f\|_{L^1}$. Notice

$$\int_{\mathbb{R}\setminus\Omega^*} |\mathcal{H}b(x)| dx \le \sum_j \int_{\mathbb{R}\setminus\Omega^*} |\mathcal{H}b_j(x)| dx$$

Since $2Q_j \subseteq \Omega^*$,

$$\int_{\mathbb{R}\setminus\Omega^*} |\mathcal{H}b(x)| dx \le \sum_j \int_{\mathbb{R}\setminus 2Q_j} |Hb_j(x)| dx.$$

Recall that each $b_j \equiv 0$ on $\left\{\mathbb{R}\setminus 2\overline{Q_j}\right\} \subseteq \{\mathbb{R}\setminus Q_j\}$ since $\int_{Q_j} b_j(y)\, dy = 0$, so we have

$$|\mathcal{H}b_j(x)| = \left|\int_{Q_j} \frac{b_j(y)}{x-y} dy\right|$$
$$= \left|\int_{Q_j} b_j(y)\left(\frac{1}{x-y} - \frac{1}{x-c_j}\right) dy\right|$$
$$\le \int_{I_j} |b_j(y)| \frac{|y-c_j|}{|x-y||x-c_j|}\, dy.$$

Because

$$|x-c_j| \le |x-y| + |y-c_j|$$
$$\le |x-y| + \frac{\ell(Q_j)}{2}$$
$$\le 2|x-y|,$$

we have $\frac{1}{|x-y|} \le \frac{2}{|x-c_j|}$, yielding

$$|\mathcal{H}b_j(x)| \le 2\int_{Q_j} |b_j(y)| \frac{|c_j - y|}{(x-c_j)^2}\, dy$$
$$\le \int_{Q_j} |b_j(y)| \frac{\ell(Q_j)}{(x-c_j)^2}\, dy.$$

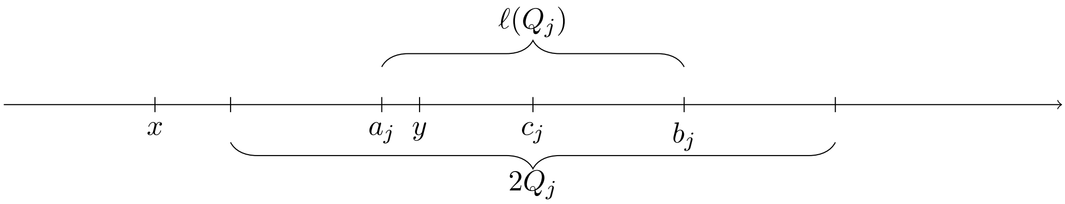

FIGURE 6. x, y and $2Q_j$

So,

$$\sum_j \int_{\mathbb{R}\setminus(2Q_j)} |\mathcal{H}b_j(x)|dx \leq \sum_j \int_{\mathbb{R}\setminus(2Q_j)} \left(\frac{\ell(Q_j)}{(x-c_j)^2} \int_{Q_j} |b_j(y)|\, dy \right) dx$$

$$= \sum_j \left[\int_{Q_j} |b_j(y)|\, dy \right] \left[\int_{\mathbb{R}\setminus(2Q_j)} \left(\frac{\ell(Q_j)}{(x-c_j)^2} \right) dx \right].$$

Notice that $\mathbb{R} \setminus (2Q_j) = (-\infty, c_j - \ell(Q_j)] \sqcup (c_j + \ell(Q_j), \infty)$, so

$$\int_{\mathbb{R}\setminus(2Q_j)} \frac{\ell(Q_j)}{(x-c_j)^2} dx = \left(\int_{-\infty}^{c_j - \ell(Q_j)} + \int_{c_j+\ell(Q_j)}^{\infty} \right) \frac{\ell(Q_j)}{(x-c_j)^2} dx$$

$$= \left[2 \cdot \frac{1}{\ell(Q_j)} \right] \cdot \ell(Q_j) = 2.$$

Therefore,

$$\sum_j \int_{\mathbb{R}\setminus(2Q_j)} |\mathcal{H}b_j(x)|dx \leq 2 \sum_j \int_{Q_j} |b_j(y)|\, dy$$

$$= 2\|b\|_{L^1}$$

$$\leq 4\|f\|_{L^1}.$$

Putting everything together, we conclude that

$$m\left(\{x : |\mathcal{H}f(x)| > \lambda\}\right) \leq \frac{18\|f\|_1}{\lambda} \quad \text{for } f \in L^1 \cap L^2.$$

□

3.5. L^p **Boundedness of the Hilbert Transform.** Recall: If $f \in L^1 \cap L^2$, then

$$m\left(\{x : |\mathcal{H}f(x)| > \lambda\}\right) \leq \frac{C\|f\|_{L^1}}{\lambda}.$$

Lemma 3.15. *Let $\{g_n\} \subseteq M$. If $\forall \lambda, \varepsilon > 0$, there exists $N = N_{\varepsilon,\lambda}$ such that $n, m > N$ implies*

$$m\left(\{x \in \mathbb{R} : |g_n(x) - g_m(x)| > \lambda\}\right) < \varepsilon,$$

i.e. $\{g_n\}$ is Cauchy in measure. Then $\exists g \in M$ such that $g_n \to g$ in measure, i.e.

$$\forall \lambda > 0, \quad \lim_{n\to\infty} m\left(\{x : |g_n(x) - g(x)| > \lambda\}\right) = 0.$$

Proof. Step (1): Find $g \in M$, $\{g_{N_k}\}$ such that $g_{N_k} \to g$ pointwise a.e. Let

$$\varepsilon = \lambda = 2^{-k}, \quad N_k > N_{2^{-k},2^{-k}},$$

and w.l.o.g.

$$N_1 < N_2 < N_3 < \ldots$$

Let

$$E_k = \left\{ x : |g_{N_{k+1}}(x) - g_{N_k}(x)| > 2^{-k} \right\},$$

then

$$\sum_{k=1}^{\infty} m(E_k) \leq \sum_{k=1}^{\infty} 2^{-k} < \infty.$$

By Borel-Cantelli Lemma, for a.e. $x \in \mathbb{R}$, $\exists K(x)$ such that $k > K(x)$ implies $x \notin E_k$. Let $\varepsilon > 0$, w.l.o.g.

$$2^{-K(x)+1} < \varepsilon.$$

If $n, m > K(x)$ (w.l.o.g. $n > m$), then

$$\begin{aligned} |g_{N_n}(x) - g_{N_m}(x)| &= \left| \sum_{k=m}^{n-1} \left(g_{N_{k+1}}(x) - g_{N_k}(x) \right) \right| \\ &\leq \sum_{k=m}^{n-1} |g_{N_{k+1}}(x) - g_{N_k}(x)| \\ &< \sum_{k=m}^{n-1} 2^{-k} \leq 2^{-m+1} \leq 2^{-kx} < \varepsilon. \end{aligned}$$

Thus, $\{g_{N_k}(x)\} \subseteq \mathbb{C}$ is Cauchy. Let $g(x) = \lim_{k\to\infty} g_{N_k}(x) \in M$.

Step (2): $g_n \to g$ in measure. Homework. □

Lemma 3.16. *Let $f \in L^1$, $\{f_n\} \subseteq L^1 \cap L^2$, $f_n \to f$ in L^1, then $\{\mathcal{H} f_n\}$ is Cauchy in measure.*

Proof. Recall that

$$m\left(\{x : |\mathcal{H} f_n(x) - \mathcal{H} f_m(x)| > \lambda\}\right) \leq \frac{C\|f_n - f_m\|_{L^1}}{\lambda}.$$

pick N such that $\|f_n - f_m\|_{L^1} < \frac{\lambda\varepsilon}{C}$, then for $n, m > N$,

$$m\left(\{x : |\mathcal{H} f_n(x) - \mathcal{H} f_m(x)| > \lambda\}\right) < \varepsilon.$$

□

Definition 3.17. Let $\widetilde{\mathcal{H}} f = \lim_{n\to\infty} \mathcal{H} f_n$, where convergence is in measure.

Remark 3.18. $\forall f \in L^1$, $m(\{x : |\widetilde{\mathcal{H}} f(x)| > \lambda\}) < \frac{2C\|f\|_1}{\lambda}$.

Remark 3.19. $\widetilde{\mathcal{H}}$ is the unique linear weak type (1,1) extension from $L^1 \cap L^2$ to L^1.

Proof. Let $\mathcal{H}' = \mathcal{H}$ on $L^1 \cap L^2$, $\mathcal{H}'$ is linear on L^1, and weak (1,1) on L^1. Let $\lambda > 0$, $f \in L^1$, we want to show

$$m(\{x : |\widetilde{\mathcal{H}} f(x) - \mathcal{H}' f(x)| > \lambda\}) = 0.$$

Let $\{f_n\} \subseteq L^1 \cap L^2$, $f_n \to f$ in L^1, notice that

$$|\widetilde{\mathcal{H}} f(x) - \mathcal{H}' f(x)| \leq |\widetilde{\mathcal{H}} f(x) - \mathcal{H} f_n(x)| + |\mathcal{H}' f_n(x) - \mathcal{H}' f(x)|.$$

Therefore,

$$\begin{aligned} m(\{x : |\widetilde{\mathcal{H}} f(x) - \mathcal{H}' f(x)| > \lambda\}) &\leq \limsup_{n\to\infty} m\left(\left\{x : |\widetilde{\mathcal{H}} f(x) - \mathcal{H} f_n(x)| > \frac{\lambda}{2}\right\}\right) \\ &\quad + \limsup_{n\to\infty} m\left(\left\{x : |\mathcal{H}'(f_n - f)(x)| > \frac{\lambda}{2}\right\}\right), \\ &\leq 0 + \limsup_{n\to\infty} C\left(\frac{2}{\lambda}\right) \|f - f_n\|_{L^1} = 0. \end{aligned}$$

□

Definition 3.20. Finally, let $\widetilde{\widetilde{\mathcal{H}}} : L^1 + L^2 \to M$ be

$$\widetilde{\widetilde{\mathcal{H}}} f = \widetilde{\mathcal{H}} f_1 + \mathcal{H} f_2,$$

where $f = f_1 + f_2$, $f_1 \in L^1, f_2 \in L^2$.

Remark 3.21. $\widetilde{\widetilde{\mathcal{H}}}$ is well-defined.

Proof. Assume $f_1 + f_2 = g_1 + g_2$, where $f_1, g_1 \in L^1$ and $f_2, g_2 \in L^2$. We want to prove

$$\widetilde{\mathcal{H}} f_1 + \mathcal{H} f_2 = \widetilde{\mathcal{H}} g_1 + \mathcal{H} g_2.$$

Notice that $f_1 - g_1 = g_2 - f_2 \in L^1 \cap L^2$, so

$$\mathcal{H}(f_1 - g_1) = \mathcal{H}(g_2 - f_2) = \mathcal{H} g_2 - \mathcal{H} f_2.$$

In addition,

$$\begin{aligned} \mathcal{H}(f_1 - g_1) &= \widetilde{\mathcal{H}}(f_1 - g_1) \\ &= \widetilde{\mathcal{H}} f_1 - \widetilde{\mathcal{H}} g_1. \end{aligned}$$

Therefore,

$$\begin{aligned} \widetilde{\mathcal{H}} f_1 - \widetilde{\mathcal{H}} g_1 &= \mathcal{H} g_2 - \mathcal{H} f_2, \\ \widetilde{\mathcal{H}} f_1 + \mathcal{H} f_2 &= \widetilde{\mathcal{H}} g_1 + \mathcal{H} g_2. \end{aligned}$$

□

Remark 3.22. By Marcinkiewicz Interpolation Theorem, $\forall p \in (1, 2)$,

$$\|\widetilde{\widetilde{\mathcal{H}}}\|_{L^p \to L^p} \le 2p^{1/p} \left[\frac{1}{p-1} + \frac{1}{2-p}\right]^{1/p} C\|f\|_{L^p}.$$

Thus for $f \in \mathcal{S}(\mathbb{R})$,

$$\|\widetilde{\mathcal{H}} f\|_{L^p} \le 2p^{1/p} \left[\frac{1}{p-1} + \frac{1}{2-p}\right]^{1/p} C\|f\|_{L^p}.$$

Remark 3.23. Homework: $\mathcal{H} f \in L^p$, if $f \in \mathcal{S}(\mathbb{R})$, $1 < p < \infty$

Theorem 3.24. *$\mathcal{H}$ extends boundedly to L^p. Also,*

$$\|\mathcal{H}\|_{L^p \to L^p} = O\left(\frac{1}{p-1}\right) \quad \text{as } p \to 1^+,$$

and

$$\|\mathcal{H}\|_{L^p \to L^p} = O(p) \quad \text{as } p \to \infty.$$

Proof. Let $2 < p < \infty$, then for $p' = \frac{p}{p-1}$, by the property of duality and adjoint operator,

$$\begin{aligned} \|\widetilde{\widetilde{\mathcal{H}}}\|_{L^p \to L^p} &= \sup_{\substack{\|g\|_{L^{p'}} = 1, \\ g \in \mathcal{S}(\mathbb{R})}} |\langle \mathcal{H} f, g\rangle_{L^2}| \\ &= \sup_{\substack{\|g\|_{L^{p'}} = 1, \\ g \in \mathcal{S}(\mathbb{R})}} |\langle \mathcal{H} g, f\rangle_{L^2}| \\ &\le \sup_{\substack{\|g\|_{L^{p'}} = 1, \\ g \in \mathcal{S}(\mathbb{R})}} \|\widetilde{\widetilde{\mathcal{H}}} g\|_{L^p} \|f\|_{L^p} \\ &\le \|\widetilde{\widetilde{\mathcal{H}}}\|_{L^{p'} \to L^{p'}} \|g\|_{L^{p'}} \|f\|_{L^p} \\ &\le \|\widetilde{\widetilde{\mathcal{H}}}\|_{L^{p'} \to L^{p'}} \|f\|_{L^p} \end{aligned}$$

$$\leq 2(p')^{1/p'} \left[\frac{1}{p'-1} + \frac{1}{2-p'} \right]^{1/p'} \|f\|_{L^p}.$$

□

Appendix A. Riesz-Thorin Interpolation

Theorem A.1 (Hadamard Three-Lines Theorem)**.** *Let* $S = \{z \in \mathbb{C} : 0 < Re z < 1\}$*. Let* f *be bounded, continuous on* $\overline{S}$*, and holomorphic on* S*. For any* $\theta \in [0,1]$*, if*

$$M_\theta = \sup_{y\in\mathbb{R}} |f(\theta + iy)|$$

then

$$M_\theta \le M_0^{1-\theta} M_1^{\theta}.$$

Proof. Take $a^z = e^{z\log a}$, consider

$$F(z) = f(z) M_0^{z-1} M_1^{-z}.$$

If M_0 or M_1 equals 0, maximum modulus principle implies that $f \equiv 0$. We may WLOG assume $M_0 = M_1 = 1$ (homework!). Thus, for $z \in \partial S$,

$$|f(z)| \le \max\{M_0, M_1\} = 1.$$

So $|f(z)| \le 1$ for $z \in S$. Let

$$K = \sup_{z\in\overline{S}} |f(z)|.$$

Consider

$$f_n(z) = \frac{f(z)}{1 + \frac{k}{n}z}, \text{ on } S_n = \{z \in S : |\text{Im}z| < n\}.$$

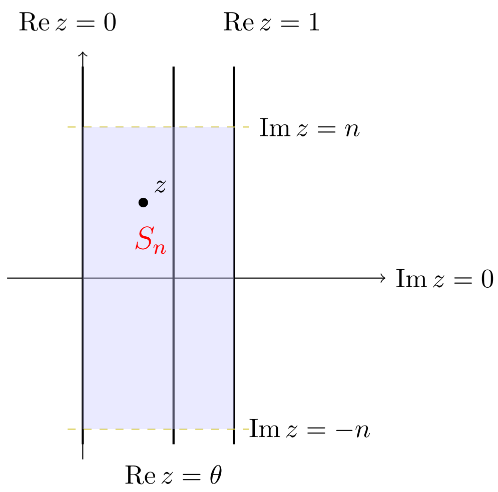

Figure 7. Three-Lines Illustration

Then, clearly, $f_n(z) \to f(z)$ as $n \to \infty$. Notice that, letting $z = x + iy \in \overline{S}$, then

$$|1 + \tfrac{k}{n}z|^2 = (1 + \tfrac{k}{n}x)^2 + \left(\tfrac{k}{n}y\right)^2 \ge 1,$$

so f_n is holomorphic on S_n, and

$$|f_n(z)| \le |f(z)| \le 1, \quad \text{if } z \in \partial S.$$

While if $z \in \partial S_n$ and $|y| = n$,

$$|1 + \tfrac{k}{n}z| \ge \left|\frac{ky}{n}\right| = k,$$

so

$$|f_n(z)| = \frac{|f(z)|}{|1 + \frac{k}{n}z|} \le \frac{k}{k} = 1.$$

Thus, we conclude that

$$|f_n(z)| \le 1 \quad \text{for } z \in \partial S_n,$$

yielding

$$|f_n(z)| \leq 1 \quad \text{for } z \in S_n,$$

and

$$|f(z)| = \lim_{n\to\infty} |f_n(z)| \leq 1 \quad \text{for } z \in S.$$

□

Theorem A.2 (Riesz-Thorin Interpolation)**.** *Let $0 < p_0, p_1 \leq \infty$, $1 \leq q_0, q_1 \leq \infty$. Fix $0 < \theta < 1$, and let*

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Let $T : F \to M$ be a linear operator such that $(Tf) \cdot g \in L^1$ for $f, g \in F$, and satisfies

$$\|Tf\|_{L^{q_0}} \leq A_0 \|f\|_{L^{p_0}}, \quad \|Tf\|_{L^{q_1}} \leq A_1 \|f\|_{L^{p_1}}, \quad \forall f \in F.$$

Then for all such f,

$$\|Tf\|_{L^{q_\theta}} \leq A_0^{1-\theta} A_1^{\theta} \|f\|_{L^{p_\theta}}.$$

Proof. Let $f \in F$. By duality, for $\frac{1}{q_\theta} + \frac{1}{q'_\theta} = 1$, $\left(L^{q_\theta}(\mathbb{R}^n)\right)^* \cong L^{q'_\theta}(\mathbb{R}^n)$. Also recall that F is dense in $L^{q'_\theta}$. Hence,

$$\|Tf\|_{L^{q_\theta}} = \sup_{\substack{g\in F \\ \|g\|_{L^{q'_\theta}}=1}} \left| \int_{\mathbb{R}^n} (Tf)(x) g(x)\, dx \right|.$$

We want to show

$$\left| \int_{\mathbb{R}^n} (Tf)(x) g(x)\, dx \right| \leq A_0^{1-\theta} A_1^{\theta} \|f\|_{L^{p_\theta}}.$$

Since both sides scale linearly when f is multiplied by a scalar, we can replace f with $f/\|f\|_{L^{p_\theta}} \in F$, it suffices to show

$$\left| \int (Tf)(x) g(x)\, dx \right| \leq A_0^{1-\theta} A_1^{\theta} \quad \text{if } \|f\|_{L^{p_\theta}} = 1.$$

Let

$$f = \sum_{j=1}^{N} a_j \alpha_j \chi_{E_j}, \quad g = \sum_{i=1}^{M} b_i \beta_i \chi_{F_i}$$

where $a_j, b_i > 0$, $\alpha_j, \beta_i \in \mathbb{C}$, and $|\alpha_j| = |\beta_i| = 1$. Define for $z = \theta + it$,

$$f_z = \sum_{j=1}^{N} a_j^{\lambda(z)} \alpha_j \chi_{E_j} \in F,$$

$$g_z = \sum_{i=1}^{M} b_i^{\mu(z)} \beta_i \chi_{F_i} \in F,$$

where

$$\lambda(z) = p_\theta \left(\frac{1-z}{p_0} + \frac{z}{p_1} \right), \quad \mu(z) = q'_\theta \left(\frac{1-z}{q'_0} + \frac{z}{q'_1} \right).$$

Define

$$I(z) = \int_{\mathbb{R}^n} Tf_z(x) \cdot g_z(x)\, dx.$$

By Morera's theorem, $I : S \to \mathbb{C}$ is holomorphic on S, bounded/continuous on $\overline{S}$. Let

$$M_\theta = \sup_{y\in\mathbb{R}} |I(\theta + iy)|.$$

Then, by Hadamard three-lines theorem

$$|I(\theta)| \leq M_\theta \leq M_0^{1-\theta} M_1^{\theta}.$$

Notice that when $z = \theta$,

$$\begin{aligned} f_\theta(x) &= \sum_{j=1}^{N} a_j^{p_\theta\left(\frac{1-\theta}{p_0}+\frac{\theta}{p_1}\right)} \alpha_j \chi_{E_j}(x) \\ &= \sum_{j=1}^{N} a_j \alpha_j \chi_{E_j}(x) \\ &= f(x), \end{aligned}$$

similarly since $\frac{1}{q'_\theta} = \frac{1-\theta}{q'_0} + \frac{\theta}{q'_1}$,

$$g_\theta(x) = g(x).$$

Therefore,

$$|I(\theta)| = \left| \int_{\mathbb{R}^n} Tf(x) \cdot g(x)\, dx \right|.$$

Let $x \in E_i$, for arbitrary $i \in \mathbb{N}$, when $\mathrm{Re} z = 0$, we have

$$\begin{aligned} |f_{0+iy}(x)| &= \left| a_j^{p_\theta\left(\frac{1-iy}{p_0}+\frac{iy}{p_1}\right)} \alpha_j \right| \\ &= a_j^{p_\theta/p_0} \\ &= |f(x)|^{p_\theta/p_0}. \end{aligned}$$

Similarly,

$$|g_{0+iy}(x)| = |g(x)|^{q'_\theta/q'_0}.$$

Summing up, when $\mathrm{Re} z = 0$,

$$\begin{aligned} |I(0+iy)| &\le \|Tf_{0+iy}\|_{L^{q_0}} \|g\|_{L^{q'_0}} \\ &\le A_0 \|f_{0+iy}\|_{L^{p_0}} \|g\|_{L^{q'_0}} \\ &= A_0 \|f\|_{L^{p_\theta}}^{p_\theta/p_0} \|g\|_{L^{q'_\theta}}^{q'_\theta/q_0} \\ &= A_0. \end{aligned}$$

Now let $x \in E_i$ as before when $\mathrm{Re} z = 1$, we have

$$\begin{aligned} |f(1+iy)| &= \left| a_j^{p_\theta\left(\frac{iy}{p_0}+\frac{1+iy}{p_1}\right)} \alpha_j \right| \\ &= a_j{}^{p_\theta/p_1} \\ &= |f(x)|^{p_\theta/p_1}, \end{aligned}$$

and similarly for g,

$$|g(1+iy)| = |g(x)|^{q'_\theta/q'_1}.$$

Thus,

$$\begin{aligned} |I(1+iy)| &\le \|Tf_{1+iy}\|_{L^{q_1}} \|g\|_{L^{q'_1}} \\ &\le A_1 \|f_{1+iy}\|_{L^{p_1}} \|g\|_{L^{q'_1}} \\ &= A_1 \|f\|_{L^{p_\theta}}^{p_\theta/p_1} \|g\|_{L^{q'_\theta}}^{q'_\theta/q'_1} \\ &= A_1. \end{aligned}$$

We conclude that

$$|I(\theta)| = \left| \int_{\mathbb{R}^n} Tf(x) \cdot g(x)\, dx \right| \le M_0^{1-\theta} M_1^{\theta} \le A_0^{1-\theta} A_1^{\theta}.$$

□

Theorem A.3 (Riesz-Thorin for $L^p \to \ell^p$)**.** *We start with a sequence analog of a simple function with support of finite measure. Let*

$$S = \{\{a_k\}_{k\in\mathbb{Z}} : a_k = 0 \textit{ for all but finitely many } k \in \mathbb{Z}\}.$$

Let $\theta, p_0, p_1, q_0, q_1, p_\theta, q_\theta$ *be as before. Let* $T : F([0,1]) \to S$ *satisfy*

$$\|Tf\|_{\ell^{q_0}(\mathbb{Z})} \leq A_0\|f\|_{L^{p_0}([0,1])},$$
$$\|Tf\|_{\ell^{q_1}(\mathbb{Z})} \leq A_1\|f\|_{L^{p_1}([0,1])}$$

for all $f \in F([0,1])$*. Then*

$$\|Tf\|_{\ell^{q_\theta}(\mathbb{Z})} \leq A_0^{1-\theta}A_1^{\theta}\|f\|_{L^{p_\theta}([0,1])}.$$

Corollary A.4 (Hausdorff–Young inequality for Fourier series)**.** *If* $1 < p \leq 2$ *and* $f \in L^p([0,1])$*, then* $\{\widehat{f}(k)\}_{k\in\mathbb{Z}} \in \ell^{p'}(\mathbb{Z})$ *and*

$$\left(\sum_{k\in\mathbb{Z}} |\hat{f}(k)|^{p'}\right)^{1/p'} \leq \|f\|_{L^p([0,1])},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

References

[1] Javier Douandikoetxea, *Fourier analysis*, AMS Graduate Studies in Mathematics vol. 29, 2001.

Department of Mathematics, Statistics, and Computer Science, University of Illinois Chicago, Chicago, IL 60607, USA

Email address: xyang212@uic.edu